

DEFORMATIONS OF ELLIPTIC CALABI–YAU MANIFOLDS

JÁNOS KOLLÁR

The aim of this note is to answer some questions about Calabi–Yau manifolds that were raised during the workshop *String Theory for Mathematicians*, which was held at the Simons Center for Geometry and Physics.

F-theory posits that the “hidden dimensions” constitute a Calabi–Yau 4-fold X that has an elliptic structure with a section. That is, there are morphisms $g : X \rightarrow B$ whose general fibers are elliptic curves and $\sigma : B \rightarrow X$ such that $g \circ \sigma = 1_B$; see [Vaf96, Don98]. In his lecture Donagi asked the following.

Question 1. Is every small deformation of an elliptic Calabi–Yau manifold also an elliptic Calabi–Yau manifold?

Question 2. Is there a good numerical characterization of elliptic Calabi–Yau manifolds?

Note that a good answer to Question 2 should give an answer to Question 1. The answers to these problems are quite sensitive to which variant of the definition of Calabi–Yau manifolds one uses. For instance, a general deformation of (Abelian variety) \times (elliptic curve) has no elliptic fiber space structure and every elliptic K3 surface has non-elliptic deformations. We prove in Section 5 that these are essentially the only such examples, even for singular Calabi–Yau varieties (31). In the smooth case, the answer is especially simple.

Theorem 3. *Let X be an elliptic Calabi–Yau manifold such that $H^2(X, \mathcal{O}_X) = 0$. Then every small deformation of X is also an elliptic Calabi–Yau manifold.*

In dimension 3 this was proved in [Wil94, Wil98].

Our results on Question 2 are less complete. Let $L_B \in H^2(B, \mathbb{Q})$ be an ample cohomology class and set $L := g^*L_B$. We interpret Question 2 to mean: *Characterize pairs (X, L) that are elliptic fiber spaces.* Following [Wil89, Ogu93], one is led to the following.

Conjecture 4. *A Calabi–Yau manifold X is elliptic iff there is a $(1, 1)$ -class $L \in H^2(X, \mathbb{Q})$ such that $(L \cdot C) \geq 0$ for every algebraic curve $C \subset X$, $(L^{\dim X}) = 0$ and $(L^{\dim X - 1}) \neq 0$.*

For threefolds, the more general results of [Ogu93, Wil94] imply Conjecture 4 if L is effective or $(L \cdot c_2(X)) \neq 0$.

As in [Wil89, Ogu93, Wil94], in higher dimensions we study the interrelation of L and of the second Chern class $c_2(X)$. By a result of [Miy88] $(L^{n-2} \cdot c_2(X)) \geq 0$ and we distinguish two (overlapping) cases.

- (Main case) If $(L^{n-2} \cdot c_2(X)) > 0$ then Conjecture 4 is solved in (11). We also check that all elliptic Calabi–Yau manifolds with a section belong to this class (47).

- (Isotrivial case) These are the examples where $X \rightarrow B$ is an analytically locally trivial fiber bundle over a dense open subset of B . An explicit construction, up-to birational equivalence, is given in (37) but I do not have a numerical characterization.

Following [Ogu93] and [MP97, Lect.10], the plan is to put both questions in the more general framework of the Abundance Conjecture [Rei83, 4.6]; see (51–52) for the precise formulation.

This approach suggests that the key is to understand the rate of growth of $h^0(X, L^m)$. If (X, L) is elliptic, then $h^0(X, L^m)$ grows like $m^{\dim X - 1}$. Given a pair (X, L) , the most important deformation-invariant quantity is the holomorphic Euler characteristic

$$\chi(X, L^m) = h^0(X, L^m) - h^1(X, L^m) + h^2(X, L^m) \dots$$

The difficulty is that in our case $h^0(X, L^m)$ and $h^1(X, L^m)$ both grow like $m^{\dim X - 1}$ and they cancel each other out. That is

$$\chi(X, L^m) = O(m^{\dim X - 2}).$$

For the main series, $\chi(X, L^m)$ does grow like $m^{\dim X - 2}$ which implies that $h^0(X, L^m)$ grows at least like $m^{\dim X - 2}$.

For the isotrivial series the order of growth of $\chi(X, L^m)$ is even smaller; in fact $\chi(X, L^m)$ can be identically zero. However, if (X, L) is elliptic, this happens only if a finite cover of X is birational to a product, so these are not particularly interesting examples.

Several of the ideas of this paper can be traced back to other sources. Sections 2–4 owe a lot to [Kaw85a, Ogu93, Wil94, Fuj11]; Sections 5–6 to [Hor76, KL09]; Sections 7–8 to [Kol93, Nak99] and to some old results of Matsusaka. Ultimately the origin of many of these methods is in the work of Kodaira on elliptic surfaces [Kod63, Sec.12]. (See [BPV84, Secs.V.7–13] for a more modern treatment.)

1. CALABI–YAU FIBER SPACES

For many reasons it is of interest to study proper morphisms with connected fibers $g' : X' \rightarrow B$ whose general fibers are birational to Calabi–Yau varieties. A special case of the Minimal Model Conjecture, proved by [Lai11, HX12], implies that every such fiber space is birational to a projective morphism with connected fibers $g : X \rightarrow B$ where X has terminal singularities and its canonical class K_X is relatively trivial, at least rationally. That is, there is a Cartier divisor F on B such that $mK_X \sim g^*F$ for some $m > 0$.

We will work with varieties with log terminal singularities, or later even with klt pairs (X, Δ) but I will state the main results for smooth varieties as well. See [KM98, Sec.2.3] for the definitions and basic properties of the singularities we use. Note also that, even if one is primarily interested in smooth Calabi–Yau varieties X , the natural setting is to allow at least canonical singularities on X and at least log terminal singularities on the base B of the elliptic fibration.

Definition 5. In this paper a *Calabi–Yau variety* is a projective variety X with log terminal singularities such that $K_X \sim_{\mathbb{Q}} 0$, that is, mK_X is linearly equivalent to 0 for some $m > 0$. Note that by [Kaw85b] this is equivalent to assuming that $(K_X \cdot C) = 0$ for every curve $C \subset X$.

Note that we allow a rather broad definition of Calabi–Yau varieties. This is very natural for algebraic geometry but less so for physical considerations.

A *Calabi–Yau fiber space* is a proper morphism with connected fibers $g : X \rightarrow B$ onto a normal variety where X has log terminal (or possibly log canonical) singularities and $K_{X_g} \sim_{\mathbb{Q}} 0$ where $X_g \subset X$ is a general fiber.

We say that $g : X \rightarrow B$ is an *elliptic* (or *Abelian* or ...) fiber space if in addition general fibers are elliptic curves (or Abelian varieties or ...). Our main interest is in the elliptic case, but in Sections 7–8 we also study the general setting.

Let X be a projective, log terminal variety and L a \mathbb{Q} -Cartier \mathbb{Q} -divisor (or divisor class) on X . We say that (X, L) is a Calabi–Yau fiber space if there is a Calabi–Yau fiber space $g : X \rightarrow B$ and an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor L_B on B such that $L \sim_{\mathbb{Q}} g^* L_B$.

In general, a divisor L is called *semi-ample* if it is the pull-back of an ample divisor by a morphism and *nef* if $(L \cdot C) \geq 0$ for every irreducible curve $C \subset X$. Every semi-ample divisor is nef but the converse usually fails. However, the hope is that for Calabi–Yau varieties nef and semi-ample are equivalent; see (51–52).

We say that a Calabi–Yau fiber space $g : X \rightarrow B$ is *relatively minimal* if $K_X \sim_{\mathbb{Q}} g^* F$ for some \mathbb{Q} -Cartier \mathbb{Q} -divisor F on B . This condition is automatic if X itself is Calabi–Yau. (These are called crepant log structures in [Kol13].)

If $g : X \rightarrow B$ is a relatively minimal Calabi–Yau fiber space and X has canonical (resp. log terminal) singularities then every other relatively minimal Calabi–Yau fiber space $g' : X' \rightarrow B$ that is birational to X also has canonical (resp. log terminal) singularities.

By [Nak88], if X has log terminal singularities then B has rational singularities, more precisely, there is an effective divisor D_B such that (B, D_B) is klt.

6 (Elliptic threefolds). Elliptic threefolds have been studied in detail. The papers [Wil89, Gra91, Nak91, Gra93, DG94, Gra94, Gro94, Wil94, Gro97, Nak02a, Nak02b, CL10, HK11, K11] give rather complete descriptions of their local and global structure. However, neither of Questions 1–2 was fully answered for threefolds.

By contrast, not even the local structure of elliptic fourfolds is understood. Double covers of the \mathbb{P}^1 -contractions described in [AW98] give some rather surprising examples; there are probably much more complicated ones as well.

Definition 7. Let $g : X \rightarrow B$ be a morphism between normal varieties. A divisor $D \subset X$ is called *horizontal* if $g(D) = B$, *vertical* if $g(D) \subset B$ has codimension ≥ 1 and *exceptional* if $g(D)$ has codimension ≥ 2 in B .

If g is birational, the latter coincides with the usual notion of exceptional divisors but the above version makes sense even if $\dim X > \dim B$. (If g is birational then there are no horizontal divisors, so this notion is not used in that case.)

8. We see in (18) that if X is smooth (or \mathbb{Q} -factorial), g is a Calabi–Yau fiber space and $D \subset X$ is exceptional then D is not g -nef. Thus, by [Lai11, HX12] the $(X, \epsilon D)$ Minimal Model Program over B (cf. [KM98, Sec.3.7]) contracts D . Thus every Calabi–Yau fiber space $g_2 : X_2 \rightarrow B_2$ is birational to a relatively minimal Calabi–Yau fiber space $g_1 : X_1 \rightarrow B_1 = B_2$ that has no exceptional divisors. Furthermore, again using [Lai11, HX12] and applying (14) it is also birational to a Calabi–Yau fiber space $g : X \rightarrow B$ where B is also \mathbb{Q} -factorial. (In general $g : X \rightarrow B$ is not unique.) Thus, in birational geometry, it is reasonable to focus on the study of

relatively minimal Calabi–Yau fiber spaces $g : X \rightarrow B$ without exceptional divisors where both X and B are \mathbb{Q} -factorial and log terminal.

From the point of view of F -theory it is especially interesting to study the examples $g' : X' \rightarrow B$ with a section $\sigma' : B \rightarrow X'$ where X' itself is Calabi–Yau. In this case the so called *Weierstrass model* is a relatively minimal model without exceptional divisors that can be explicitly constructed as follows.

Let L_B be an ample divisor on B . Then $\sigma'(B) + mg'^*L_B$ is nef and big on X' , hence a large multiple of it is base point free (cf. [KM98, Sec.3.2]). This gives a morphism $h : X' \rightarrow X$ where X is still Calabi–Yau (usually with canonical singularities) and $g : X \rightarrow B$ has a section $\sigma : B \rightarrow X$ whose image is g -ample. Thus every fiber of g has dimension 1 and so $g : X \rightarrow B$ has no exceptional divisors.

Furthermore, $R^1h_*\mathcal{O}_{X'} = 0$ which implies that every deformation of X' comes from a deformation of X ; see (54).

The next result says that once $g : X \rightarrow B$ looks like a relatively minimal Calabi–Yau fiber space outside a subset of codimension ≥ 2 then it is a relatively minimal Calabi–Yau fiber space.

Proposition 9. *Let $g : X \rightarrow B$ be a projective fiber space with X log terminal. Assume the following.*

- (1) *There are no g -exceptional divisors (7).*
- (2) *There is a closed subset $Z \subset B$ of codimension ≥ 2 such that K_X is numerically trivial on the fibers over $B \setminus Z$.*
- (3) *B is \mathbb{Q} -factorial.*

Then $g : X \rightarrow B$ is a relatively minimal Calabi–Yau fiber space.

Proof. First note that, as a very special case of (14), there is a \mathbb{Q} -Cartier \mathbb{Q} -divisor F_1 on $B \setminus Z$ such that

$$K_X|_{X \setminus g^{-1}(Z)} \sim_{\mathbb{Q}} g^*F_1.$$

Since B is \mathbb{Q} -factorial, F_1 extends to a \mathbb{Q} -Cartier \mathbb{Q} -divisor F on B .

Thus every point $b \in B$ has an open neighborhood $b \in U_b \subset B$ and an integer $m_b > 0$ such that

$$\mathcal{O}_X(m_b K_X)|_{g^{-1}(U_b \setminus Z)} \cong g^*\mathcal{O}_{U_b}(m_b F|_{U_b}) \cong g^*\mathcal{O}_{U_b} \cong \mathcal{O}_{g^{-1}(U_b)}.$$

By (1), $g^{-1}(Z)$ has codimension ≥ 2 in $g^{-1}(U_b)$ and hence the constant 1 section of $\mathcal{O}_{g^{-1}(U_b \setminus Z)}$ extends to a global section of $\mathcal{O}_X(m_b K_X)|_{g^{-1}(U_b)}$ that has neither poles nor zeros. Thus

$$\mathcal{O}_X(m_b K_X)|_{g^{-1}(U_b)} \cong \mathcal{O}_{g^{-1}(U_b)}.$$

Since this holds for every $b \in B$, we conclude that $K_X \sim_{\mathbb{Q}} g^*F$. □

2. THE MAIN CASE

The next theorem gives a characterization of the main series of elliptic Calabi–Yau fiber spaces. (For the log version see (55).) The proof is quite short but it relies on auxiliary results that are proved in the next two sections.

Theorem 10. *Let X be a projective variety of dimension n with log terminal singularities and L a Cartier divisor on X . Assume that K_X is nef and $(L^{n-2} \cdot \text{td}_2(X)) > 0$ where $\text{td}_2(X)$ is the second Todd class of X (24). Then (X, L) is a relatively minimal, elliptic fiber space iff*

- (1) L is nef,
- (2) $L - \epsilon K_X$ is nef for $0 \leq \epsilon \ll 1$,
- (3) $(L^n) = (L^{n-1} \cdot K_X) = 0$ and
- (4) (L^{n-1}) is nonzero in $H^{2n-2}(X, \mathbb{Q})$.

Note that if (X, L) is a relatively minimal, elliptic fiber space then L is semi-ample (5) and, as we see in (13), the only hard part of (10) is to show that conditions (10.1–4) imply that L is semi-ample. In particular, (10) also holds over fields that are not algebraically closed.

This immediately yields the following partial answer to Question 1.

Corollary 11. *Let X be a smooth, projective variety of dimension n and L a Cartier divisor on X . Assume that $K_X \sim_{\mathbb{Q}} 0$ and $(L^{n-2} \cdot c_2(X)) > 0$. Then (X, L) is an elliptic fiber space iff*

- (1) L is nef,
- (2) $(L^n) = 0$ and
- (3) (L^{n-1}) is nonzero in $H^{2n-2}(X, \mathbb{Q})$.

Definition 12. Let Y be a projective variety and D a Cartier divisor on X . If $m > 0$ is sufficiently divisible, then, up-to birational equivalence, the map given by global sections of $\mathcal{O}_Y(mD)$

$$Y \dashrightarrow I(Y, D) \stackrel{\text{bir}}{\sim} I_m(Y, D) \hookrightarrow \mathbb{P}(H^0(Y, \mathcal{O}_Y(mD))) \quad \text{is independent of } m.$$

It is called the *Iitaka fibration* of (Y, D) . The *Kodaira dimension* of D (or of (Y, D)) is $\kappa(D) = \kappa(Y, D) := \dim I(Y, D)$.

If D is nef, the *numerical dimension* of D (or of (Y, D)), denoted by $\nu(D)$ or $\nu(Y, D)$, is the largest natural number r such that the self-intersection $(D^r) \in H^{2r}(Y, \mathbb{Q})$ is nonzero. Equivalently, $(D^r \cdot H^{n-r}) > 0$ for some (or every) ample divisor H .

It is easy to see that $\kappa(D) \leq \nu(D)$. This was probably first observed by Matsusaka as a corollary of his theory of variable intersection cycles; see [Mat72] or [LM75, p.515].

13 (Proof of (10)). First note that $\kappa(L) \geq n - 2$ by (25). We will also need this for some perturbations of L .

Set $L_m := L - \frac{1}{m}K_X$. For $m \gg 1$ we see that L_m is nef, $(L_m^{n-2} \cdot \text{td}_2(X)) > 0$ and (L_m^{n-1}) is nonzero in $H^{2n-2}(X, \mathbb{Q})$. Note that $mL = K_X + mL_m$ hence

$$m^n(L^n) = \sum_{i=0}^n m^{n-i} (K_X^i \cdot L_m^{n-i}).$$

Since K_X and L_m are both nef, all the terms on the right hand side are ≥ 0 . Their sum is zero by assumption, hence $(K_X^i \cdot L_m^{n-i}) = 0$ for every i . Thus (25) also applies to L_m and we get that $\kappa(L_m) \geq n - 2$.

We can now apply (15) with $\Delta = 0$ and $D := 2mL_m$ and $K_X + 2mL_m = 2mL_{2m}$ to conclude that $\nu(L_m) \leq \kappa(L_{2m})$. Since we know that $\nu(L_m) = \dim X - 1$ we conclude that $\kappa(L_{2m}) = \dim X - 1$.

Finally use (14) with $S = (\text{point})$, $2mL$ instead of L and $a = 1$ to obtain that some multiple of L is semi-ample. That is, there is a morphism with connected fibers $g : X \rightarrow B$ and an ample \mathbb{Q} -divisor L_B such that $L \sim_{\mathbb{Q}} g^*L_B$. Note that $(L^{\dim B}) \neq 0$ but $(L^{\dim B+1}) = 0$ so comparing with (3–4) we see that $\dim B = \dim X - 1$. By the adjunction formula, the canonical class of the general fiber is proportional to $(L^{n-1} \cdot K_X) = 0$, thus $g : X \rightarrow B$ is an elliptic fiber space. \square

We have used the following theorem due to [Kaw85a] and [Fuj11].

Theorem 14. *Let (X, Δ) be an irreducible, projective, klt pair and $g : X \rightarrow S$ a morphism with generic fiber X_g . Let L be a \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . Assume that*

- (1) *L and $L - K_X - \Delta$ are g -nef and*
- (2) *$\nu((L - K_X - \Delta)|_{X_g}) = \kappa((L - K_X - \Delta)|_{X_g}) = \nu(((1+a)L - K_X - \Delta)|_{X_g}) = \kappa(((1+a)L - K_X - \Delta)|_{X_g})$ for some $a > 0$.*

Then there is a factorization $g : X \xrightarrow{h} B \xrightarrow{\pi} S$ and a π -ample \mathbb{Q} -Cartier \mathbb{Q} -divisor L_B on B such that $L \sim_{\mathbb{Q}} h^ L_B$.* \square

3. ADJOINT SYSTEMS OF LARGE KODAIRA DIMENSION

The following is modeled on [Ogu93, 2.4].

Proposition 15. *Let (X, Δ) be a projective, klt pair such that $K_X + \Delta$ is pseudo-effective, that is, its cohomology class is a limit of effective classes. Let D be an effective, nef, \mathbb{Q} -Cartier \mathbb{Q} -divisor on X such that $\kappa(K_X + \Delta + D) \geq \dim X - 2$. Then $\nu(D) \leq \kappa(K_X + \Delta + D)$.*

Proof. There is nothing to prove if $\kappa(K_X + \Delta + D) = \dim X$. Thus assume that $\kappa(K_X + \Delta + D) \leq \dim X - 1$ and let $g : X \dashrightarrow B$ be the Iitaka fibration (cf. [Laz04, 2.1.33]). After some blow-ups we may assume that g is a morphism and X, B are smooth.

The generic fiber of g is a smooth curve or surface (S, Δ_S) such that $K_S + \Delta_S$ is pseudo-effective. Since abundance holds for curves and surfaces [Kol92, Sec.11], this implies that $\kappa(K_S + \Delta_S) \geq 0$. Furthermore, by Iitaka's theorem (cf. [Laz04, 2.1.33]) $\kappa(K_S + \Delta_S + D|_S) = 0$.

If D is disjoint from S then, by (17.2), $\nu(D) \leq \dim B = \kappa(K_X + \Delta + D)$ and we are done. Otherwise $D|_S$ is an effective, nonzero, nef divisor on S . We obtain a contradiction by proving that $\kappa(K_S + \Delta_S + D|_S) \geq 1$.

If S is a curve, then $\deg D|_S > 0$ hence $\kappa(K_S + \Delta_S + D|_S) \geq \kappa(D|_S) = 1$. If S is a surface, then $\kappa(K_S + \Delta_S + D|_S) \geq 1$ is proved in (16). \square

Lemma 16. *Let (S, Δ_S) be a projective, klt surface such that $\kappa(K_S + \Delta_S) \geq 0$. Let D be a nonzero, effective, nef \mathbb{Q} -divisor. Then $\kappa(K_S + \Delta_S + D) \geq 1$.*

Proof. Since $\kappa(K_S + \Delta_S + D) \geq \kappa(K_S + \Delta_S)$ we only need to consider the case when $\kappa(K_S + \Delta_S) = 0$. Let $\pi : (S, \Delta_S) \rightarrow (S^m, \Delta_S^m)$ be the minimal model. It is obtained by repeatedly contracting curves that have negative intersection number with $K_S + \Delta_S$. These curves also have negative intersection number with $K_S + \Delta_S + \epsilon D$ for $0 < \epsilon \ll 1$. Thus

$$\pi : (S, \Delta_S + \epsilon D) \rightarrow (S^m, \Delta_S^m + \epsilon D^m)$$

is also the minimal model and $(S^m, \Delta_S^m + \epsilon D^m)$ is klt for $0 < \epsilon \ll 1$. By the Hodge index theorem, every effective divisor contracted by π has negative self-intersection, thus D^m is again a nonzero, effective, nef \mathbb{Q} -divisor.

Since abundance holds for klt surface pairs (cf. [Kol92, Sec.11]), we see that $K_{S^m} + \Delta_S^m \sim_{\mathbb{Q}} 0$ and $\kappa(K_{S^m} + \Delta_S^m + \epsilon D^m) \geq 1$. Since D is effective, we obtain that $\kappa(K_S + \Delta_S + D) \geq \kappa(K_S + \Delta_S + \epsilon D) = \kappa(K_{S^m} + \Delta_S^m + \epsilon D^m) \geq 1$. \square

Lemma 17. *Let $g : X \rightarrow B$ be a proper morphism with connected general fiber X_g . Let D be an effective, nef, \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . Then*

- (1) *either $D|_{X_g}$ is a nonzero nef divisor*
- (2) *or D is disjoint from X_g and $(D^{\dim B+1}) = 0$. Thus $\nu(D) \leq \dim B$.*

Proof. We are done if $D|_{X_g}$ is nonzero. If it is zero then D is vertical hence there is an ample divisor L_B such that $g^*L_B \sim D + E$ where E is effective. Then

$$(g^*L_B^r) - (D^r) = \sum_{i=0}^{r-1} (E \cdot g^*L_B^i \cdot D^{r-1-i})$$

shows that $(D^r) \leq (g^*L_B^r)$. Since $((g^*L_B)^{\dim B+1}) = g^*(L_B^{\dim B+1}) = 0$, we conclude that $(D^{\dim B+1}) = 0$. \square

Lemma 18. *Let $g : X \rightarrow B$ be a proper morphism with connected fibers and D an effective, exceptional, \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . Then D is not g -nef.*

Proof. Let $|H|$ be a very ample linear system on X and $S \subset X$ the intersection of $\dim X - 2$ general members of $|H|$. Then $g|_S : S \rightarrow B$ is generically finite over its image and $D \cap S$ is $g|_S$ -exceptional. By the Hodge index theorem we conclude that $(D^2 \cdot H^{\dim X-2}) < 0$, a contradiction. \square

4. ASYMPTOTIC ESTIMATES FOR COHOMOLOGY GROUPS

19. Let X be a smooth variety and $g : X \rightarrow B$ a Calabi-Yau fiber space of relative dimension m over a smooth curve B . Assume that $K_{X_g} \sim 0$ where X_g denotes a general fiber. It is easy to see that the sheaves $R^m g_* \mathcal{O}_X$ and $g_* \omega_{X/B}$ are line bundles and dual to each other. For elliptic surfaces these sheaves were computed by Kodaira. His results were clarified and extended to higher dimensions by [Fuj78]. We will need the following consequences of their results.

The degree $\deg g_* \omega_{X/B}$ is ≥ 0 and it can be written as a sum of 2 terms. One is a global term (determined by the j -invariant of the fibers in the elliptic case) which is zero iff $g : X \rightarrow B$ is *generically isotrivial*, that is, g is an analytically locally trivial fiber bundle over a dense open set $B^0 \subset B$. The other is a local term, supported at the points where the local monodromy of the local system $R^m g_* \mathbb{Q}_{X^0}$ is nontrivial. There is a precise formula for the local term, but we only need to understand what happens with generically isotrivial families. For these the local term is positive iff the local monodromy has eigenvalue $\neq 1$ on $g_* \omega_{X^0/B^0} \subset \mathcal{O}_{B^0} \otimes_{\mathbb{Q}} R^m g_* \mathbb{Q}_{X^0}$.

Over higher dimensional bases, $R^m g_* \mathcal{O}_X$ and $g_* \omega_{X/B}$ are rank 1 sheaves, and the above considerations describe their codimension 1 behavior. In particular, we see the following.

- (1) $c_1(g_* \omega_{X/B})$ is linearly equivalent to a sum of effective \mathbb{Q} -divisors. It is zero only if $g : X \rightarrow B$ is isotrivial over a dense open set B^0 and the local monodromy around each irreducible component of $B \setminus B^0$ has eigenvalue $= 1$ on $g_* \omega_{X^0/B^0} \subset \mathcal{O}_{B^0} \otimes_{\mathbb{Q}} R^m g_* \mathbb{Q}_{X^0}$.
- (2) $c_1(R^m g_* \mathcal{O}_X) = -c_1(g_* \omega_{X/B})$.

Frequently $c_1(g_* \omega_{X/B})$ is denoted by $\Delta_{X/B}$.

Corollary 20. *Let $g : X \rightarrow B$ be an elliptic fiber space of dimension n and L a line bundle on B . Then*

$$\begin{aligned} \chi(X, g^* L^m) &= \frac{(L^{n-2} \cdot \Delta_{X/B})}{2(n-2)!} m^{n-2} + O(m^{n-3}) \quad \text{and} \\ h^i(X, g^* L^m) &= O(m^{n-3}) \quad \text{for } i \geq 2. \end{aligned}$$

Proof. By the Leray spectral sequence,

$$\chi(X, g^* L^m) = \sum (-1)^i \chi(B, L^m \otimes R^i g_* \mathcal{O}_X).$$

For $i \geq 2$ the support of $R^i g_* \mathcal{O}_X$ has codimension ≥ 2 in B , hence its cohomologies contribute only to the $O(m^{n-3})$ term.

Since g has connected fibers, $g_* \mathcal{O}_X \cong \mathcal{O}_B$ and $c_1(R^1 g_* \mathcal{O}_X) \sim_{\mathbb{Q}} -\Delta_{X/B}$ by (19.2). We conclude by applying (23) to both terms. \square

21. Similar formulas apply to arbitrary Calabi–Yau fiber spaces $g : X \rightarrow B$ with general fiber F . For $m \gg 1$ we have

$$H^i(X, g^* L^m) = H^0(B, L^m \otimes R^i g_* \mathcal{O}_X) = \chi(B, L^m \otimes R^i g_* \mathcal{O}_X). \quad (21.1)$$

Setting $k = \dim B$, (23) computes $H^i(X, g^* L^m)$ as

$$\frac{m^k}{k!} h^i(F, \mathcal{O}_F)(L^k) + \frac{m^{k-1}}{(k-1)!} \left(L^{k-1} \cdot (c_1(R^i g_* \mathcal{O}_X) - \frac{h^i(F, \mathcal{O}_F)}{2} K_B) \right) + O(m^{k-2}).$$

These imply that

$$\chi(X, g^* L^m) = \chi(F, \mathcal{O}_F) \cdot \frac{m^k}{k!} (L^k) + O(m^{k-1}). \quad (21.2)$$

If $\chi(F, \mathcal{O}_F) \neq 0$ then this describes the asymptotic behavior of $\chi(X, g^* L^m)$. However, if $\chi(F, \mathcal{O}_F) = 0$, which happens for Abelian fibers, then we have to look at the next term which gives that

$$\chi(X, g^* L^m) = \frac{m^{k-1}}{(k-1)!} \left(L^{k-1} \cdot \sum_{i=1}^{\dim F} (-1)^i c_1(R^i g_* \mathcal{O}_X) \right) + O(m^{k-2}). \quad (21.3)$$

If F is an elliptic curve then the sum on the right hand side has only 1 nonzero term. For higher dimensional Abelian fibers there are usually several nonzero terms and sometimes they cancel each other.

This is one reason why elliptic fibers are easier to study than higher dimensional Abelian fibers. The other difficulty with higher dimensional fibers is that the Euler characteristic only tells us that $h^0 + h^2 + h^4 + \dots$ grows as expected. Proving that $h^0 \neq 0$ would need additional arguments.

The next result, while stated in all dimensions, is truly equivalent to Kodaira’s formula [BPV84, V.12.2].

Corollary 22. *Let $g : X \rightarrow B$ be a relatively minimal elliptic fiber space of dimension n and L a line bundle on B . Then $(L^{n-2} \cdot \Delta_{X/B}) = (g^* L^{n-2} \cdot \text{td}_2(X))$.*

Proof. Expanding the Riemann–Roch formula $\chi(X, L) = \int_X \text{ch}(L) \cdot \text{td}(X)$ gives that

$$\chi(X, L^m) = \frac{(L^n)}{n!} \cdot m^n - \frac{(L^{n-1} \cdot K_X)}{2(n-1)!} \cdot m^{n-1} + \frac{(L^{n-2} \cdot \text{td}_2(X))}{(n-2)!} \cdot m^{n-2} + O(m^{n-3}).$$

Comparing this with (20) yields the claim. \square

We used several versions of the asymptotic Riemann–Roch formula.

Lemma 23. *Let X be a normal, projective variety of dimension n , L a line bundle on X and F a coherent sheaf that is locally free in codimension 1. Then*

$$\chi(X, L^m \otimes F) = \frac{(L^n) \cdot \text{rank } F}{n!} m^n + \frac{(L^{n-1} \cdot (c_1(F) - \frac{\text{rank } F}{2} K_X))}{(n-1)!} m^{n-1} + O(m^{n-2}). \quad \square$$

24 (Riemann–Roch with rational singularities). The Todd classes of a singular variety X are not always easy to compute, but if X has rational singularities then there is a straightforward formula in terms of the Chern classes of any resolution $h : X' \rightarrow X$.

By definition, rational singularity means that $R^i h_* \mathcal{O}_{X'} = 0$ for $i > 0$. Thus $\chi(X, L) = \chi(X', h^* L)$ for any line bundle L on X . By the projection formula this implies that $\chi(X, L) = \int_X \text{ch}(L) \cdot h_* \text{td}(X')$ and in fact $\text{td}(X) = h_* \text{td}(X')$ (cf. [Ful98, Thm.18.2]). In particular we see that the second Todd class of X is

$$\text{td}_2(X) = h_* \left(\frac{c_1(X')^2 + c_2(X')}{12} \right).$$

The following numerical version of (20) was used in the proof of (10).

Lemma 25. *Let X be a normal, projective variety of dimension n . Let L be a nef line bundle on X such that $(L^n) = (L^{n-1} \cdot K_X) = 0$ but $(L^{n-1}) \neq 0$. Then*

$$h^0(X, L^m) - h^1(X, L^m) = \frac{(L^{n-2} \cdot \text{td}_2(X))}{(n-2)!} \cdot m^{n-2} + O(m^{n-3}).$$

Proof. The assumptions $(L^n) = (L^{n-1} \cdot K_X) = 0$ imply that the right hand side equals $\chi(X, L^m)$. Thus the equality follows if $h^i(X, L^m) = O(m^{n-3})$ for $i \geq 2$. The latter is a special case of (26). \square

Lemma 26. *Let X be a projective variety of dimension n and F a torsion free coherent sheaf on X . Let L be a nef line bundle on X and set $d = \nu(X, L)$. Then*

$$\begin{aligned} h^i(X, F \otimes L^m) &= O(m^d) \quad \text{for } i = 0, \dots, n-d \text{ and} \\ h^{n-j}(X, F \otimes L^m) &= O(m^{j-1}) \quad \text{for } j = 0, \dots, d-1. \end{aligned}$$

Note the key feature of the estimate: the order of growth of H^i is m^d for $i \leq n-d$, then for $i = n-d+1$ it drops by 2 to m^{d-2} and then it drops by 1 for each increase of i . This strengthens [Laz04, 1.4.40] but the proof is essentially the same.

Proof. We use induction on $\dim X$. By Fujita’s theorem (cf. [Laz04, 1.4.35]) we can choose a general very ample divisor A on X such that

$$h^i(X, F \otimes \mathcal{O}_X(A) \otimes L^m) = 0 \quad \text{for all } i \geq 1 \text{ and } m \geq 1.$$

We get an exact sequence

$$0 \rightarrow F \otimes L^m \rightarrow F \otimes \mathcal{O}_X(A) \otimes L^m \rightarrow G \otimes L^m \rightarrow 0$$

where G is a torsion free coherent sheaf on A . For $i \geq 1$ its long cohomology sequence gives surjections (even isomorphisms for $i \geq 2$)

$$H^{i-1}(A, G \otimes L^m) \rightarrow H^i(X, F \otimes L^m).$$

By induction this shows the claim except for $i = 0$.

One can realize F as a subsheaf of a sum of line bundles, thus it remains to prove that $H^0(X, F \otimes L^m) = O(m^d)$ when $F \cong \mathcal{O}_X(H)$ is a very ample line bundle. The exact sequence

$$0 \rightarrow L^m \rightarrow \mathcal{O}_X(H) \otimes L^m \rightarrow \mathcal{O}_H(H|_H) \otimes L^m \rightarrow 0$$

finally reduces the problem to $\kappa(L) \leq \nu(L)$ which was discussed in (12). \square

5. DEFORMING MORPHISMS

Here we answer Question 1 but first two technical issues need to be discussed: the distinction between étale and quasi-étale covers and the existence of non-Calabi–Yau deformations. Both appear only for singular Calabi–Yau varieties.

Definition 27. Following [Cat07], a finite morphism $\pi : U \rightarrow V$ is called *quasi-étale* if there is a closed subvariety $Z \subset V$ of codimension ≥ 2 such that π is étale over $V \setminus Z$.

If V is a normal variety, then there is a one-to-one correspondence between quasi-étale covers of V and finite, étale covers of $V \setminus \text{Sing } V$.

In particular, if X is a Calabi–Yau variety then there is a quasi-étale morphism $X_1 \rightarrow X$ such that $K_{X_1} \sim 0$.

Among all such covers $X_1 \rightarrow X$ there is a unique smallest one, called the *index 1 cover* of X , which is Galois with cyclic Galois group. We denote it by $X^{\text{ind}} \rightarrow X$.

28 (Deformation theory). For a general introduction, see [Har10]. By a deformation of a proper scheme (or analytic space) X we mean a flat, proper morphism $g : \mathbf{X} \rightarrow (0 \in S)$ to a pointed scheme (or analytic space) together with a fixed isomorphism $X_0 \cong X$.

By a deformation of a morphism of proper schemes (or analytic spaces) $f : X \rightarrow Y$ we mean a morphism $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ where \mathbf{X} is a deformation of X , \mathbf{Y} is a deformation of Y and $\mathbf{f}|_{X_0} = f$.

When we say that an assertion holds for all *small deformations* of X , this means that for every deformation $g : \mathbf{X} \rightarrow (0 \in S)$ there is an étale (or analytic) neighborhood $(0 \in S') \rightarrow (0 \in S)$ such that the assertion holds for $g' : \mathbf{X} \times_S S' \rightarrow (0 \in S')$.

29 (Deformations of Calabi–Yau varieties). Let X be a Calabi–Yau variety. If X is smooth (or has canonical singularities, or K_X is Cartier) then every small deformation of X is again a Calabi–Yau variety. This, however, fails in general; see (48) for an example where X is a surface with quotient singularities.

Dealing with such unexpected deformations is a basic problem in the moduli theory of higher dimensional varieties; see [Kol12, Sec.4], [HK10, Sec.14B] or [AH11] for a discussion and solutions. For Calabi–Yau varieties one can use a global trivialization of the canonical bundle to get a much simpler answer.

We say that a deformation $g : \mathbf{X} \rightarrow (0 \in S)$ of X over a reduced, local space S is a *Calabi–Yau deformation* if the following equivalent conditions hold:

- (1) Every fiber of g is a Calabi–Yau variety.
- (2) The deformation can be lifted to a deformation $g^{\text{ind}} : \mathbf{X}^{\text{ind}} \rightarrow (0 \in S)$ of X^{ind} , the index 1 cover of X .

Thus studying Calabi–Yau deformations of Calabi–Yau varieties is equivalent to studying deformations of Calabi–Yau varieties whose canonical class is Cartier. As we noted, for the latter every deformation is automatically a Calabi–Yau deformation. Thus we do not have to deal with this issue at all.

Theorem 30. *Let X be a Calabi–Yau variety and $g : X \rightarrow B$ an elliptic fiber space. Then at least one of the following holds.*

- (1) *The morphism g extends to every small Calabi–Yau deformation of X .*
- (2) *There is a quasi-étale cover $\tilde{X} \rightarrow X$ such that the Stein factorization $\tilde{g} : \tilde{X} \rightarrow \tilde{B}$ of $\tilde{X} \rightarrow B$ is one of the following*

- (a) $(\tilde{g} : \tilde{X} \rightarrow \tilde{B}) \cong (p_1 : \tilde{B} \times (\text{elliptic curve}) \rightarrow \tilde{B})$ where p_1 is the first projection or
 (b) $(\tilde{g} : \tilde{X} \rightarrow \tilde{B}) \cong (p_1 : \tilde{Z} \times (\text{elliptic K3}) \rightarrow \tilde{Z} \times \mathbb{P}^1)$
 where \tilde{Z} is a Calabi-Yau variety of dimension $\dim X - 2$ and p_1 is the product of the first projection with the elliptic pencil map of the K3 surface.

Proof. As noted in (29), we may assume that $K_X \sim 0$.

By [KMM92] there is a unique map (up-to birational equivalence) $h : B \dashrightarrow Z$ whose general fiber F is smooth, proper, rationally connected and whose target Z is not uniruled by [GHS03]. (See [Kol96, Chap.IV] for a detailed treatment or [AK03] for an introduction.) Next apply [KL09, Thm.3] to $X \dashrightarrow Z$ to conclude that there is a finite étale cover $\tilde{X} \rightarrow X$, a product decomposition $\tilde{X} \cong Y \times \tilde{Z}$ and a generically finite map $\tilde{Z} \dashrightarrow Z$ that factors $\tilde{X} \dashrightarrow Z$.

If $\dim Z = \dim B$ then we are in case (2.a). If $\dim Z = \dim B - 1$ then the generic fiber of $\tilde{B} \rightarrow \tilde{Z}$ is \mathbb{P}^1 . Furthermore, $\dim Y = 2$, hence either Y is an elliptic K3 surface and we are in case (2.b) or Y is an Abelian surface that has an elliptic pencil and after a further cover we are again in case (2.a).

It remains to prove that if $\dim F \geq 2$ then the assertion of (1) holds. By (35) it is sufficient to check that

$$\mathrm{Hom}_B(\Omega_B, R^1 g_* \mathcal{O}_X) = 0.$$

To see this, note that (19) and $\omega_X \sim \mathcal{O}_X$ imply that $R^1 g_* \mathcal{O}_X \cong (g_* \omega_{X/B})^{-1} \cong \omega_B$. Therefore

$$\mathrm{Hom}_B(\Omega_B, R^1 g_* \mathcal{O}_X) \cong \mathrm{Hom}_B(\Omega_B, \omega_B) \cong (\Omega_B^{\dim B - 1})^{**}$$

where $()^{**}$ denotes the double dual or reflexive hull. By taking global sections we get that

$$\mathrm{Hom}_B(\Omega_B, R^1 g_* \mathcal{O}_X) = H^0(B, (\Omega_B^{\dim B - 1})^{**}).$$

Let $B' \rightarrow B$ be a resolution of singularities and $F' \subset B'$ a general fiber of $B' \dashrightarrow Z$. Since F' is rationally connected, if $C \subset F'$ is a general rational curve then

$$T_{F'}|_C \cong \sum \mathcal{O}_C(a_i) \quad \text{where } a_i > 0 \ \forall i;$$

see [Kol96, IV.3.9]. Thus $T_{B'}|_C$ is a sum of line bundles $\mathcal{O}_C(a_i)$ where $a_i > 0$ for $\dim F$ summands and $a_i = 0$ for the rest. Since $\dim F \geq 2$ we conclude that

$$\wedge^{\dim B - 1} T_{B'}|_C \cong \sum \mathcal{O}_C(b_i) \quad \text{where } b_i > 0 \text{ for every } i.$$

By duality this gives that $H^0(B', \Omega_{B'}^{\dim B - 1}) = 0$. Finally we use that B has log terminal singularities by [Nak88] and so [GKKP11] shows that

$$\mathrm{Hom}_B(\Omega_B, R^1 g_* \mathcal{O}_X) = H^0(B, (\Omega_B^{\dim B - 1})^{**}) = H^0(B', \Omega_{B'}^{\dim B - 1}) = 0. \quad \square$$

We are now ready to answer Question 1.

Theorem 31. *Let X be an elliptic Calabi-Yau variety such that $H^2(X, \mathcal{O}_X) = 0$. Then every small Calabi-Yau deformation of X is also an elliptic Calabi-Yau variety.*

Proof. Let $g : X \rightarrow B$ be an elliptic Calabi-Yau variety. By (30) every small Calabi-Yau deformation of X is also an elliptic Calabi-Yau variety except possibly when there is a quasi-étale cover $\tilde{X} \rightarrow X$ such that

- (1) either $\tilde{X} \cong \tilde{Z} \times (\text{elliptic curve})$

(2) or $\tilde{X} \cong \tilde{Z} \times (\text{elliptic K3})$.

In both cases, \tilde{X} can have non-elliptic deformations but we show that these do not correspond to a deformation of X . Here we use that $H^2(X, \mathcal{O}_X) = 0$.

Let $\pi : \mathbf{X} \rightarrow (0 \in S)$ be a flat deformation of X over a local scheme S . Let L be the pull-back of an ample line bundle from B to X . Since $H^2(X, \mathcal{O}_X) = 0$, L lifts to a line bundle \mathbf{L} on \mathbf{X} (cf. [Gro62, p.236-16]) thus we get a line bundle $\tilde{\mathbf{L}}$ on $\tilde{\mathbf{X}}$. We need to show that a large multiple of \mathbf{L} is base-point-free over S ; then it gives the required morphism $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{B}$. One can check base-point-freeness of some multiple after a finite surjection, thus it is enough to show that some multiple of $\tilde{\mathbf{L}}$ is base-point-free over S .

The first case (more generally, deformations of products with Abelian varieties) is treated in (41).

In the K3 case note first that every small deformation of \tilde{X} is of the form $\tilde{\mathbf{Z}} \times_S \tilde{\mathbf{F}}$ where $\tilde{\mathbf{F}} \rightarrow S$ is a flat family of K3 surfaces. This is a trivial case of (35); see (54) for an elementary argument. Hence we only need to show that the restriction of $\tilde{\mathbf{L}}$ to $\tilde{\mathbf{F}}$ is base-point-free over S . Equivalently, that the elliptic structure of the central K3 surface \tilde{F} is preserved by our deformation. The restriction of $\tilde{\mathbf{L}}$ to every fiber of $\tilde{\mathbf{F}} \rightarrow S$ gives a nonzero, nef line bundle with self-intersection 0, hence an elliptic pencil. \square

32 (Deformation of sections). Let $g : X \rightarrow B$ be an elliptic Calabi–Yau fiber space with a section $S \subset X$. Let us assume first that S is a Cartier divisor in X . (This is automatic if X is smooth.) Then S is g -nef, g -big and $S \sim_{\mathbb{Q}, g} K_X + S$ hence $R^i g_* \mathcal{O}_X(S) = 0$ for $i > 0$; cf. [KM98, Sec.2.5]. Thus $H^i(X, \mathcal{O}_X(S)) = H^i(B, g_* \mathcal{O}_X(S))$ for every i . In order to compute $g_* \mathcal{O}_X(S)$ we use the exact sequence

$$0 \rightarrow \mathcal{O}_B = g_* \mathcal{O}_X \xrightarrow{\alpha} g_* \mathcal{O}_X(S) \rightarrow g_* \mathcal{O}_S(S|_S)$$

A degree 1 line bundle over an elliptic curve has only 1 section, thus α is an isomorphism over an open set where the fiber is a smooth elliptic curve. Since $g_* \mathcal{O}_S(S|_S) \cong \mathcal{O}_S(S|_S)$ is torsion free we conclude that $g_* \mathcal{O}_X(S) \cong \mathcal{O}_B$. Thus

$$H^1(X, \mathcal{O}_X(S)) = H^1(B, \mathcal{O}_B) \subset H^1(X, \mathcal{O}_X).$$

If $H^2(X, \mathcal{O}_X) = 0$ then the line bundle $\mathcal{O}_X(S)$ lifts to every small deformation of X and if $H^1(X, \mathcal{O}_X) = 0$ then the unique section of $\mathcal{O}_X(S)$ also lifts.

The situation is quite different if the section is not assumed Cartier. For instance, let $X_0 \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a general hypersurface of multidegree $(3, 3)$ containing $S := \mathbb{P}^2 \times \{p\}$ for some point p . Then X_0 is a Calabi–Yau variety and the first projection shows that it is elliptic with a section. Note that X_0 is singular, it has 9 ordinary nodes along S .

By contrast, if $X_t \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a smooth hypersurface of multidegree $(3, 3)$ then the restriction map $\text{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) \rightarrow \text{Pic}(X_t)$ is an isomorphism by the Lefschetz hyperplane theorem. Thus the degree of every divisor $D \subset X_t$ on the general fiber of the first projection $X_t \rightarrow \mathbb{P}^2$ is a multiple of 3. Therefore $X_t \rightarrow \mathbb{P}^2$ does not even have a rational section.

As an aside, we consider the general question of deforming morphisms $g : X \rightarrow Y$ whose target is not uniruled.

There are some obvious examples when not every deformation of X gives a deformation of $g : X \rightarrow Y$. For example, let A_1, A_2 be positive dimensional Abelian

varieties and $g : A_1 \times A_2 \rightarrow A_2$ the second projection. A general deformation of $A_1 \times A_2$ is a simple Abelian variety which has no maps to lower dimensional Abelian varieties. One can now get more complicated examples by replacing $A_1 \times A_2$ by say a complete intersection subvariety or by a cyclic cover. The next result says that this essentially gives all examples.

Theorem 33. *Let X be a projective variety with rational singularities, Y a normal variety and $g : X \rightarrow Y$ a surjective morphism with connected fibers. Assume that Y is not uniruled. Then at least one of the following holds.*

- (1) *Every small deformation of X gives a deformation of $(g : X \rightarrow Y)$.*
- (2) *There is a quasi-étale cover $\tilde{Y} \rightarrow Y$, a smooth variety Z and positive dimensional Abelian varieties A_1, A_2 such that the lifted morphism $\tilde{g} : \tilde{X} := X \times_Y \tilde{Y} \rightarrow \tilde{Y}$ factors as*

$$\begin{array}{ccc} \tilde{X} & \rightarrow & Z \times A_2 \times A_1 \\ \tilde{g} \downarrow & & \downarrow \\ \tilde{Y} & \cong & Z \times A_2. \end{array}$$

Proof. By (35) every deformation of X gives a deformation of $g : X \rightarrow Y$ if

$$\mathrm{Hom}_Y(\Omega_Y, R^1 g_* \mathcal{O}_X) = 0. \quad (33.3)$$

Thus we need to show that if (33.3) fails then we get a structural description as in (33.2).

Let $C \subset Y$ be a very general complete intersection curve. Since Y is not uniruled, $\Omega_Y|_C$ is semi-positive by [Miy88]; see also [Kol92, Sec.9].

Let $Y^0 \subset Y$ be a dense open set and $X^0 := g^{-1}(Y^0)$ such that $g^0 : X^0 \rightarrow Y^0$ is smooth. Set $C^0 := X^0 \cap C$. By [Ste76], $(R^1 g_* \mathcal{O}_X)|_C$ is the (lower) canonical extension of the top quotient of the variation of Hodge structures $R^1 g_*^0 \mathbb{Q}_{X^0}|_{C^0}$. (Note that [Ste76] works with ω_{X^0/Y^0} but the proof is essentially the same; see [Kol86b, pp.177–179].) Thus $(R^1 g_* \mathcal{O}_X)|_C$ is semi-negative by [Ste76]. Moreover, the part that is not strictly negative corresponds to a variation of sub-Hodge structures that is a direct summand and becomes trivial after a suitable quasi-étale cover $Y_1 \rightarrow Y$ [Del71, Thm.4.2.6]. This direct summand corresponds to a direct factor of the Albanese variety of $X_1 := Y_1 \times_Y X$, giving the Abelian variety A_1 .

Once the flat part of $R^1 g_*^0 \mathbb{Q}_{X^0}$ is trivial,

$$\mathrm{Hom}_Y(\Omega_Y, R^1 g_* \mathcal{O}_X)|_C \cong (T_Y \otimes R^1 g_* \mathcal{O}_X)|_C$$

has a global section iff $T_Y|_C$ has a global section. Thus $H^0(Y, T_Y) \neq 0$ and so $\dim \mathrm{Aut}(Y) > 0$. Since Y is not uniruled, $\mathrm{Aut}^0(Y)$ has no linear algebraic subgroups, thus the connected component $\mathrm{Aut}^0(Y)$ is an Abelian variety A_2 . By (34), A_2 becomes a direct factor after a suitable étale cover $\tilde{Y} \rightarrow Y_1 \rightarrow Y$. \square

The following result was essentially known to [Ser01, Ses63]; see [Bri10] for the general theory.

Proposition 34. *Let W be a normal, projective variety and A an Abelian variety acting faithfully on W . Then there is a normal, projective variety Z and an A -equivariant étale morphism $A \times Z \rightarrow W$.*

Proof. Let $T \subset W$ be the generic orbit. The quotient $V := W/A$ exists; it is the normalization of the closure of the point $[T]$ corresponding to T in the Chow variety $\mathrm{Chow}(W)$. Since $W \rightarrow V$ is a generically isotrivial A -bundle, using (44), we obtain

that $W \cong (\tilde{V} \times A)/G$ where $\tilde{V} \rightarrow V$ is a finite (ramified cover) and G acts faithfully on \tilde{V} and on A . Since the A -action descends to W , the G -action on A commutes with translations. An automorphism of an Abelian variety that commutes with all translations is itself a translation. Thus G acts on A via translations and so the G -action on $\tilde{V} \times A$ is fixed point free. Therefore $\tilde{V} \times A \rightarrow W$ is étale. \square

The following is a combination of [Hor76, Thm.8.1] and the method of [Hor76, Thm.8.2] in the smooth case and [BHPS12, Prop.3.10] in general.

Theorem 35. *Let $f : X \rightarrow Y$ be a morphism of proper varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $\mathrm{Hom}_Y(\Omega_Y, R^1f_*\mathcal{O}_X) = 0$.*

Then for every small deformation of \mathbf{X} of X there is a small deformation \mathbf{Y} of Y such that f lifts to $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$. \square

Note that if X is smooth or, more generally, if it has rational singularities, then $R^1f_*\mathcal{O}_X$ is a reflexive sheaf by [Kol86a, 7.8]. Thus, if Y is normal then

$$\mathrm{Hom}_Y(\Omega_Y, R^1f_*\mathcal{O}_X) = \mathrm{Hom}_{Y^{\mathrm{ns}}}(\Omega_{Y^{\mathrm{ns}}}, R^1f_*\mathcal{O}_X|_{Y^{\mathrm{ns}}})$$

where $Y^{\mathrm{ns}} \subset Y$ is the smooth locus. Therefore we can check the vanishing of $\mathrm{Hom}_Y(\Omega_Y, R^1f_*\mathcal{O}_X) = 0$ by finding general projective curves $C \subset Y^{\mathrm{ns}}$ such that the vector bundle $(T_Y \otimes R^1f_*\mathcal{O}_X)|_C$ has no global sections.

6. SMOOTHINGS OF VERY SINGULAR VARIETIES

One can frequently construct smooth varieties by first exhibiting some very singular, even reducible schemes with suitable numerical invariants and then smoothing them. For such Calabi–Yau examples see [KN94]. Thus it is of interest to know when an elliptic fiber space structure is preserved by a smoothing. In some cases, when (31) does not apply, the following result, relying on (11), provides a quite satisfactory answer.

Proposition 36. *Let X be a projective, Gorenstein scheme of pure dimension n such that ω_X is numerically trivial and $H^2(X, \mathcal{O}_X) = 0$. Let $g : X \rightarrow B$ be a morphism whose general fibers (over every irreducible component of B) are curves of arithmetic genus 1. Assume also that every irreducible component of X dominates an irreducible component of B .*

*Let L_B be an ample line bundle on B and assume that $\chi(X, g^*L_B^m)$ is a polynomial of degree $\dim X - 2$. Then every smoothing of X is an elliptic fiber space.*

Warning. Note that we do not claim that g lifts to every deformation of X . In the example (50) X has smoothings, which are elliptic, and also other singular deformations that are not elliptic.

Proof. As before, $H^2(X, \mathcal{O}_X) = 0$ implies that g^*L_B lifts to every small deformation [Gro62, p.236–16]. Thus we have a deformation $h : (\mathbf{X}, \mathbf{L}) \rightarrow (0 \in S)$ of $(X_0, L_0) \cong (X, L = g^*L_B)$.

We claim that \mathbf{L} is h -nef and $K_{\mathbf{X}}$ is trivial on the fibers of h . This is a somewhat delicate point since being nef is not known to be an open condition in general. We go around this problem as follows.

Let $(X_{\mathrm{gen}}, L_{\mathrm{gen}})$ be a generic fiber. (Note the difference between generic and general.) First we show that L_{gen} is nef and $K_{X_{\mathrm{gen}}} \sim_{\mathbb{Q}} 0$. Indeed, assume that $(L_{\mathrm{gen}} \cdot C_{\mathrm{gen}}) < 0$ for some curve C_{gen} . Let $C_0 \subset X_0$ be a specialization of C_{gen} .

Then $(L_0 \cdot C_0) = (L_{gen} \cdot C_{gen}) < 0$ gives a contradiction. A similar argument shows that $(K_{X_{gen}} \cdot C_{gen}) = 0$ for every curve C_{gen} .

Next, the deformation invariance of $\chi(X, g^* L_B^m)$ and Riemann–Roch (cf. (22) and (24)) show that

$$(L_{gen}^{n-2} \cdot c_2(X_{gen})) = (n-2)! \cdot (\text{coefficient of } m^{n-2} \text{ in } \chi(X, g^* L_B^m)).$$

Therefore $(L_{gen}^{n-2} \cdot c_2(X_{gen})) > 0$ and, as we noted after (10), this implies that $|mL_{gen}|$ is base point free for some $m > 0$.

Thus there is a dense Zariski open subset $S^0 \subset S$ such that $|mL_s|$ is base point free for $s \in S^0$, hence (X_s, L_s) is an elliptic fiber space for $s \in S^0$. We repeat the argument for the generic points of $S \setminus S^0$ and conclude by Noetherian induction. \square

Note that the universal deformation space of a proper scheme can be represented by a scheme [Art69], thus the above argument takes care of analytic deformations as well. It may be useful, however, to see how to modify the proof to work directly in the analytic case when there are no generic points.

The (Barlet or Douady) space of curves in $h : \mathbf{X} \rightarrow (0 \in S)$ has only countably many irreducible components, thus there are countably many closed subspaces $S_i \subsetneq S$ such that every curve $C_s \subset X_s$ is deformation equivalent to a curve $C_0 \subset X_0$. In particular, L_s is nef and $K_{X_s} \sim_{\mathbb{Q}} 0$ whenever $s \notin \cup S_i$. Thus (X_s, L_s) is an elliptic fiber space for $s \notin \cup S_i$.

By semicontinuity, there are closed subvarieties $T_m \subsetneq S$ such that

$$h_* \mathcal{O}_{\mathbf{X}}(m\mathbf{L}) \otimes \mathbb{C}_s = H^0(X_s, \mathcal{O}_{X_s}(mL_s)) \quad \text{for } s \notin T_m.$$

Thus if $s \notin \cup_i S_i \cup \cup_m T_m$ and $\mathcal{O}_{X_s}(m_0 L_s)$ is generated by global sections then

$$\phi_{m_0} : h^*(h_* \mathcal{O}_{\mathbf{X}}(m_0 \mathbf{L})) \rightarrow \mathcal{O}_{\mathbf{X}}(m_0 \mathbf{L})$$

is surjective along X_s . Thus there is a dense Zariski open subset $S^0 \subset S$ such that ϕ_{m_0} is surjective for all $s \in S^0$. Now we can finish by Noetherian induction as before.

7. CALABI–YAU ORBIBUNDLES

The techniques of this section are mostly taken from [Kol93, Sec.6] and [Nak99].

Definition 37. A Calabi–Yau fiber space $g : X \rightarrow B$ is called an *orbibundle* if it can be obtained by the following construction.

Let \tilde{B} be a normal variety, F a Calabi–Yau variety and $\tilde{X} := \tilde{B} \times F$. Let G be a finite group, $\rho_B : G \rightarrow \text{Aut}(\tilde{B})$ and $\rho_F : G \rightarrow \text{Aut}(F)$ two faithful representations. Set

$$(g : X \rightarrow B) := (\tilde{X}/G \rightarrow \tilde{B}/G);$$

it is a generically isotrivial Calabi–Yau fiber space.

(It would seem more natural to require the above property only locally on B . We see in (44) that in the algebraic case the two versions are equivalent. However, if X is a Kähler manifold, then the local and global versions are different.)

For any non-ruled variety Z , the connected component $\text{Aut}^0(Z)$ of $\text{Aut}(Z)$ is an Abelian variety, its elements are called translations. The quotient $\text{Aut}(Z)/\text{Aut}^0(Z)$ is the discrete part of the automorphism group.

For G acting on F , let $G_t := \rho_F^{-1} \text{Aut}^0(F) \subset G$ be the normal subgroup of translations and set $X^d := \tilde{X}/G_t$. Then $G_d := G/G_t$ acts on X^d and $X = X^d/G_d$. Thus every orbibundle comes with 2 covers:

$$\begin{array}{ccccc} X & \xleftarrow{\tau_X} & X^d & \xleftarrow{\pi_X} & \tilde{X} \\ g \downarrow & & g^d \downarrow & & \tilde{g} \downarrow \\ B & \xleftarrow{\tau_B} & B^d & \xleftarrow{\pi_B} & \tilde{B} \end{array} \quad (37.1)$$

We see during the proof of (44) that the cover $X \leftarrow X^d$ corresponding to the discrete part of the monodromy representation is uniquely determined by $g : X \rightarrow B$. By contrast, the $X^d \leftarrow \tilde{X}$ part is not unique. Its group of deck transformations is $G_t \subset \text{Aut}^0(F)$, hence Abelian. It is not even clear that there is a natural “smallest” choice of $X^d \leftarrow \tilde{X}$.

If $F = A$ is an Abelian variety then $g^d : X^d \rightarrow B^d$ is a Seifert bundle where an orbibundle $g^s : X^s \rightarrow B^s$ is called a *Seifert bundle* if $F = A$ is an Abelian variety and G acts on A by translations. Note that in this case the A -action on $\tilde{B} \times A$ descends to an A -action on X^s and $B^s = X^s/A$. Thus the reduced structure of every fiber is a smooth Abelian variety isogenous to A .

Lemma 38. *Notation as above. Then*

- (1) π_X and τ_X are étale in codimension 1 (that is, quasi-étale),
- (2) π_X and τ_X are étale in codimension 2 if one of the following holds
 - (a) G acts freely on F outside a codimension ≥ 2 subset or
 - (b) $K_F \sim 0$ and $\Delta_{X/B} = 0$.

Proof. The first claim is clear since both ρ_F, ρ_B are faithful.

Since ρ_F, ρ_B are faithful, τ_X fails to be étale in codimension 2 iff some $1 \neq g \in G$ fixes a divisor $\tilde{D}_B \subset \tilde{B}$ and also a divisor $D_F \subset F$. This is excluded by (2.a).

Next we check that (2.b) implies (2.a). At a general point $p \in D_F$ choose local g -equivariant coordinates x_1, \dots, x_m such that $D_F = (x_1 = 0)$. Thus $\rho_F(g)^*$ acts on x_1 non-trivially but it fixes x_2, \dots, x_m . Let ω_0 be a nonzero section of ω_F . Locally near p we can write

$$\omega_0 = f \cdot dx_1 \wedge \cdots \wedge dx_m,$$

thus $\rho_F(g)^*$ acts on $H^0(F, \omega_F)$ with the same eigenvalue as on x_1 .

Thus, by (19.1), the image of \tilde{D}_X gives a positive contribution to $\Delta_{X/B}$. This contradicts $\Delta_{X/B} = 0$. \square

There are some obvious deformations of X obtained by deforming \tilde{B} and F in a family $\{(\tilde{B}_t, F_t)\}$ such that the representations ρ_B, ρ_F lift to $\rho_{B,t} : G \rightarrow \text{Aut}(\tilde{B}_t)$ and $\rho_{F,t} : G \rightarrow \text{Aut}(F_t)$.

In general, not every deformation of X arises this way. For instance, let \tilde{B} and $F = A$ be elliptic curves and X the Kummer surface of $\tilde{B} \times A$. The obvious deformations of X form a 2-dimensional family obtained by deforming \tilde{B} and A . Thus a general deformation of X is not obtained this way and it is not even elliptic. Even worse, a general elliptic deformation of X is also not Kummer, thus not every deformation of the morphism $(g : X \rightarrow B)$ is obtained by the quotient construction.

Theorem 39. *Let $g : X \rightarrow B$ be a Calabi–Yau orbibundle with general fiber F . Assume that X has log terminal singularities, $H^2(X, \mathcal{O}_X) = 0$, $\kappa(X) \geq 0$, $K_F \sim 0$ and $\Delta_{X/B} = 0$.*

Then every flat deformation of X arises from a flat deformation of $(\tilde{B}, F, \rho_B, \rho_F)$.

Proof. Let L_B be an ample line bundle on B and set $L := g^*L_B$.

Let $h : \mathbf{X} \rightarrow (0 \in S)$ be a deformation of $X_0 \cong X$. In the sequel we will repeatedly replace S by a smaller analytic (or étale) neighborhood of 0 if necessary.

Since $H^2(X, \mathcal{O}_X) = 0$, L lifts to a line bundle \mathbf{L} on \mathbf{X} by [Gro62, p.236-16].

Since $K_F \sim 0$ and $\Delta_{X/B} = 0$, (38) implies that $\pi : \tilde{X} \rightarrow X$ is étale in codimension 2. Thus, by [Kol95, Thm.12], the cover π lifts to a cover $\Pi : \tilde{\mathbf{X}} \rightarrow \mathbf{X}$.

Finally we show that the product decomposition $\tilde{X} \cong \tilde{B} \times F$ lifts to a product decomposition

$$\tilde{\mathbf{X}} \cong \tilde{\mathbf{B}} \times_S \mathbf{F}$$

where $\tilde{\mathbf{B}} \rightarrow S$ is a flat deformation of \tilde{B} and $\mathbf{F} \rightarrow S$ is a family of Calabi–Yau varieties over S . After a further étale cover of $\tilde{F} \rightarrow F$ we may assume that $\tilde{F} \cong Z \times A$ where $H^1(Z, \mathcal{O}_Z) = 0$ and A is an Abelian variety. Set $\hat{X} := \tilde{B} \times Z \times A$; then $\hat{X} \rightarrow \tilde{X}$ lifts to a deformation $\hat{\mathbf{X}} \rightarrow \tilde{\mathbf{X}} \rightarrow S$.

First we use (41) to show that the product decomposition $\hat{X} \cong (\tilde{B} \times Z) \times A$ lifts to a product decomposition

$$\hat{\mathbf{X}} \cong \widehat{\mathbf{BZ}} \times_S \mathbf{A}$$

where $\widehat{\mathbf{BZ}} \rightarrow S$ is a flat deformation of $\tilde{B} \times Z$ and $\mathbf{A} \rightarrow S$ is a family of Abelian varieties over S . There are two separate issues here: we have to make sure that automorphisms of \hat{X} lift to automorphisms of $\hat{\mathbf{X}}$ and we have to ensure that the lifted A -action does not get mixed-up with the possible automorphisms of \tilde{B} .

The deformation of the product $\tilde{B} \times Z$ is much easier; we discuss it in (54). \square

40. In some sense, elliptic curves give the only examples of Calabi–Yau orbibundles that have a non-obvious deformation.

Assume that F has no finite étale cover \tilde{F} that can be written as a product $\tilde{F} \cong F_1 \times E$ where E is an elliptic curve. We claim that if $H^2(X, \mathcal{O}_X) = 0$ then every flat deformation of X arises from a flat deformation of $(\tilde{B}, F, \rho_B, \rho_F)$.

To see this consider the relative Albanese

$$\begin{array}{ccc} X^0 & \xrightarrow{alb} & \text{Alb}_{B^0}(X^0) \\ \downarrow & & \downarrow \\ B^0 & = & B^0. \end{array}$$

If an automorphism of an Abelian variety A fixes a divisor then that divisor is an Abelian subvariety and A has an elliptic curve factor up-to isogeny. Thus (38) shows that our arguments above apply to $\text{Alb}_{B^0}(X^0) \rightarrow B^0$. Hence by taking a suitable cover that is étale outside a codimension ≥ 2 subset, we can trivialize $\text{Alb}_{B^0}(X^0) \rightarrow B^0$. Thus we may assume that $\text{Alb}_{B^0}(X^0) \cong B^0 \times \text{Alb}(F)$.

The Albanese map $F \rightarrow \text{Alb}(F)$ is a fiber bundle, thus after taking a finite cover $\text{Alb}'(F) \rightarrow \text{Alb}(F)$ we get $F' \rightarrow F$ such that $F' \cong F_1 \times \text{Alb}'(F)$. If $H^1(F_1, \mathcal{O}_{F_1}) \neq 0$ then $\dim \text{Alb}(F') > \dim \text{Alb}(F)$ and we repeat the above argument.

Thus eventually we get a cover $\hat{X} \rightarrow X$ that is étale outside a codimension ≥ 2 subset such that $\hat{g} : \hat{X} \rightarrow B$ is an orbibundle with fiber \hat{F} and there is a morphism $\hat{q} : \hat{X} \rightarrow B \times \text{Alb}(\hat{F})$ which is an orbibundle with fiber G with $H^1(G, \mathcal{O}_G) = 0$.

By [Kol95, Thm.12], every deformation $h : \mathbf{X} \rightarrow (0 \in S)$ of X lifts to a deformation $\hat{\mathbf{X}}$ of \hat{X} . Since $H^2(X, \mathcal{O}_X) = 0$, we can lift L to a line bundle \mathbf{L} on \mathbf{X} , hence to a line bundle $\hat{\mathbf{L}}$ on $\hat{\mathbf{X}}$. We see in (54) that \hat{g} lifts to a morphism $\hat{\mathbf{g}} : \hat{\mathbf{X}} \rightarrow \mathbf{BA}$

where \mathbf{BA} is a deformation of $B \times \text{Alb}(\hat{F})$. We can use $\hat{\mathbf{g}}$ to descend $\hat{\mathbf{L}}$ to a line bundle $\hat{\mathbf{L}}_{BA}$ on \mathbf{BA} . Now we can use (41) to see that $\mathbf{BA} \cong \mathbf{B} \times_S \mathbf{A}$ where \mathbf{B} is a deformation of B and \mathbf{A} is a deformation of $\text{Alb}(\hat{F})$. \square

Lemma 41. *Let $Y \rightarrow S$ be a flat, proper morphism whose fibers are normal and L a line bundle on Y . Let $0 \in S$ be a point such that*

- (1) Y_0 is not birationally ruled,
- (2) an Abelian variety $A_0 \subset \text{Aut}^0(Y_0)$ acts faithfully on Y_0 ,
- (3) L_0 is nef, L_0 is numerically trivial on the A_0 -orbits but not numerically trivial on general A'_0 -orbits for any $A_0 \subsetneq A'_0 \subset \text{Aut}^0(Y_0)$.

Then, possibly after shrinking S , there is an Abelian scheme $A \rightarrow S$ extending A_0 such that A acts faithfully on Y .

Proof. By [Mat68, p.217] (see also [Kol85, p.392]) possibly after shrinking S , $g^a : \text{Aut}^0(Y/S) \rightarrow S$ is a smooth Abelian scheme, where $\text{Aut}^0(Y/S)$ denotes the identity component of the automorphism scheme $\text{Aut}(Y/S)$. Working étale locally, we may assume that there is a section $Z \subset Y$. Acting on Z gives a morphism $\rho_Z : \text{Aut}^0(Y/S) \rightarrow Y$. Then $\rho_Z^* L$ is a nef line bundle on $\text{Aut}^0(Y/S)$. The kernel of the cup-product map

$$c_1(\rho_Z^* L) : R^1 g_*^a \mathbb{Q} \rightarrow R^3 g_*^a \mathbb{Q}$$

is a variation of sub-Hodge structures, hence it corresponds to a smooth Abelian subfamily $A \subset \text{Aut}^0(Y/S)$. By (3), this is the required extension of A_0 .

The quotient then exists by [Ses63]. \square

We will also need to understand the class group of an orbibundle.

42 (Divisors on orbibundles). We use the notation of (37) and of (43).

By [BGS11, 5.3], (see also [HK11, CL10] for the elliptic case) the class group of $\tilde{B} \times F$ is

$$\text{Cl}(\tilde{B} \times F) = \text{Cl}(\tilde{B}) + \text{Cl}(F) + \text{Hom}(\text{Alb}^{\text{rat}}(\tilde{B}), \text{Pic}^0(F)). \quad (42.1)$$

This comes with a natural G -action and, up-to torsion, the class group of the quotient is

$$\text{Cl}(B) + \text{Cl}(F)^G + \text{Hom}(\text{Alb}^{\text{rat}}(\tilde{B}), \text{Pic}^0(F))^G. \quad (42.2)$$

If \tilde{B} has rational singularities then $\text{Alb}^{\text{rat}}(\tilde{B}) = \text{Alb}(\tilde{B})$ and then the extra component $\text{Hom}(\text{Alb}(\tilde{B}), \text{Pic}^0(F))$ gives \mathbb{Q} -Cartier divisors.

We will use the following variant of this.

Claim 42.3. Let $g : X \rightarrow B$ be an orbibundle such that X has log terminal singularities. Then the natural map

$$\text{Cl}(B)/\text{Pic}(B) + (\text{Cl}(F)/\text{Pic}(F))^G \rightarrow \text{Cl}(X)/\text{Pic}(X)$$

is an isomorphism modulo torsion. In particular, if B and F are \mathbb{Q} -factorial then so is X .

Proof. By (38), $\tau_X : X^d \rightarrow X$ is étale in codimension 1, hence X^d also has log terminal singularities. As noted in (5), this implies that B^d has rational singularities.

Let us now study more carefully the right hand side of (42.2). Let $G_t \subset G$ denote the subgroup of translations. Then

$$\text{Hom}(\text{Alb}^{\text{rat}}(\tilde{B}), \text{Pic}^0(F))^G \subset \text{Hom}(\text{Alb}^{\text{rat}}(\tilde{B}), \text{Pic}^0(F))^{G_t}.$$

Since translations act trivially on $\text{Pic}^0(F)$, the latter can be identified (up-to torsion) as

$$\begin{aligned} \text{Hom}(\text{Alb}^{\text{rat}}(\tilde{B}), \text{Pic}^0(F))^{G_t} \otimes \mathbb{Q} &\cong \text{Hom}(\text{Alb}^{\text{rat}}(\tilde{B})^{G_t}, \text{Pic}^0(F)) \otimes \mathbb{Q} \\ &\cong \text{Hom}(\text{Alb}^{\text{rat}}(B^d), \text{Pic}^0(F)) \otimes \mathbb{Q} \\ &\cong \text{Hom}(\text{Alb}(B^d), \text{Pic}^0(F)) \otimes \mathbb{Q}. \end{aligned}$$

Thus this extra term gives only \mathbb{Q} -Cartier divisors on X^d and hence also on X . \square

The following local example shows that it is not enough to assume that B has rational singularities. Set $\tilde{B} = (u^3 + v^3 + w^3 = 0) \subset \mathbb{A}^3$ and $E = (x^3 + y^3 + z^3 = 0) \subset \mathbb{P}^2$. On both factors, $\mathbb{Z}/3$ acts by weights $(0, 0, 1)$. Then $B = \tilde{B}/\frac{1}{3}(0, 0, 1) \cong \mathbb{A}^2$ is even smooth but

$$X = \tilde{B} \times E/\frac{1}{3}(0, 0, 1) \times (0, 0, 1)$$

is not \mathbb{Q} -factorial. For instance, the closure of the graph of the natural projection $\tilde{B} \dashrightarrow E$ gives a non- \mathbb{Q} -Cartier divisor on X .

Definition 43 (Albanese varieties). For a smooth projective variety V let $\text{Alb}(V)$ denote the Albanese variety, that is, the target of the universal morphism from V to an Abelian variety. (See [BPV84, Sec.I.13] or [Gro62, p.236-16] for introductions.)

There are 2 ways to generalize this concept to normal varieties.

The above definition yields what we again call the *Albanese variety* $\text{Alb}(V)$. Alternatively, the *rational Albanese variety* $\text{Alb}^{\text{rat}}(V)$ is defined as the target of the universal rational map from V to an Abelian variety. One can identify $\text{Alb}^{\text{rat}}(V) = \text{Alb}(V')$ where $V' \rightarrow V$ is any resolution of singularities.

It is easy to see that if V has log terminal (more generally rational) singularities then $\text{Alb}^{\text{rat}}(V) = \text{Alb}(V)$.

8. GENERICALLY ISOTRIVIAL CALABI–YAU FIBER SPACES

In this section we prove that all generically isotrivial Calabi–Yau fiber spaces are essentially Calabi–Yau orbibundles.

Theorem 44. *Let $g : X \rightarrow B$ be a projective, generically isotrivial, Calabi–Yau fiber space.*

- (1) *There is a unique Calabi–Yau orbibundle $(g^{\text{orb}} : X^{\text{orb}} \rightarrow B)$ that is birational to $g : X \rightarrow B$.*
- (2) *X is isomorphic to X^{orb} if the following hold*
 - (a) *X is \mathbb{Q} -factorial and log terminal,*
 - (b) *$g : X \rightarrow B$ is relatively minimal and has no exceptional divisors,*
 - (c) *B is \mathbb{Q} -factorial.*

Proof. Let $B^0 \subset B$ be a Zariski open subset over which $X^0 \rightarrow B^0$ is isotrivial with general fiber F . This gives a well-defined representation

$$\rho : \pi_1(B^0) \rightarrow \text{Aut}(F)/\text{Aut}^0(F).$$

Let $B^{(d,0)} \rightarrow B^0$ be the corresponding étale, Galois cover with group G_d and $B^d \rightarrow B$ its extension to a (usually ramified) Galois cover of B with group G_d . This gives the well-defined cover in (37.1).

The trivialization of the translation part is more subtle and it depends on additional choices.

A general $\text{Aut}^0(F)$ -orbit $A_F \subset F$ defines an isotrivial Abelian family $X^{(d,0)} \supset A_X^{(d,0)} \rightarrow B^{(d,0)}$. By assumption there is a g -ample line bundle L on X . It pulls back to a relatively ample line bundle L_A on $A_X^{(d,0)}$. We may assume that its degree on the general fiber is at least 3. Let $T^{(d,0)} \subset A_X^{(d,0)}$ be the subscheme as in (45). Since L_A is G_d -invariant, $T^{(d,0)}$ is G_d -equivariant hence it defines a monodromy representation of $\pi_1(B^0) \rightarrow \text{Aut}(F)$; let G denote its image.

Let $\tilde{B}^0 \rightarrow B^0$ be the corresponding étale, Galois cover with group G and $\tilde{B} \rightarrow B$ its extension to a (usually ramified) Galois cover of B with group G .

By pull-back we obtain an isotrivial, Abelian fiber space $\tilde{A}_X^0 \rightarrow \tilde{B}^0$ with a trivialization of the m -torsion points. For $m \geq 3$ this implies that $\tilde{A}_X^0 \cong \tilde{B}^0 \times A$. (This is quite elementary, cf. [ACG11, p.513].) Thus the same pull-back also trivializes $X^0 \rightarrow B^0$. We can compactify \tilde{X}^0 as $\tilde{X} := \tilde{B} \times A$.

The G -action on $\tilde{A}_X^0 \cong \tilde{B}^0 \times A$ can be given as

$$g : (\tilde{b}, a) \mapsto (\rho_B(g) \cdot \tilde{b}, \rho_{A,\tilde{b}}(g) \cdot a).$$

Note that $\rho_{A,\tilde{b}}$ preserves the m -torsion points and the automorphisms of an Abelian torsor that preserve any finite nonempty set form a discrete group. Thus in fact $\rho_{A,\tilde{b}}$ is independent of \tilde{b} and hence the G -action on \tilde{X} is given by

$$g : (\tilde{b}, a) \mapsto (\rho_B(g) \cdot \tilde{b}, \rho_A(g) \cdot a)$$

for some isomorphism $\rho_B : G \cong \text{Gal}(\tilde{B}/B)$ and homomorphism $\rho_B : G \rightarrow \text{Aut}(F)$. We can replace \tilde{B} by $\tilde{B}/\ker \rho_B$, hence we may assume that ρ_B is faithful. By construction X is birational to $X^{\text{orb}} := \tilde{X}/G$.

In general, birational maps between relatively minimal models are very special. First there are divisorial contractions along which the canonical class is trivial. In our case these are excluded by (2.a). In the non- \mathbb{Q} -factorial case there could be small contractions, but X^{orb} is also \mathbb{Q} -factorial by (42.3).

Finally there can be flops, but the orbifold does not have any suitable extremal rays by (46). Thus X is isomorphic to X^{orb} if the conditions (2.a–c) hold. \square

45 (Multisections of Abelian families). Let E be a smooth projective curve of genus 1 and L a line bundle of degree m on E . If $m = 1$ then L has a unique section, thus we can associate a point $p \in E$ to L . If $m \geq 2$, then sections define a linear equivalence class $|L|$ of m points. If we fix a point $0 \in E$ to be the origin, then we can add these m points together and get a well defined point of E associated to L . This, however, depends on the choice of the origin.

To get something invariant, let us look at the points $p \in E$ such that $m \cdot p \in |L|$. There are m^2 such points, together forming a translate of the subgroup of m -torsion points. This construction also works in families.

Let $g : X \rightarrow B$ be a smooth, projective morphism whose fibers E_b are curves of genus 1. Let L be a line bundle on X that has degree m on each fiber. Then there is a closed subscheme $T \subset X$ such that $g|_T : T \rightarrow B$ is étale of degree m^2 and every fiber $T_b \subset E_b$ is a translate of the subgroup of m -torsion points.

There is a similar construction for higher dimensional Abelian varieties. For clarity, I say *Abelian torsor* when talking about an Abelian variety without a specified origin.

Thus let A be an Abelian torsor of dimension d and L an ample line bundle on A . It has a first Chern class $\tilde{c}_1(L)$ in the Chow group and we get $\tilde{c}_1(L)^d$ as an

element of the Chow group of 0-cycles. (It is important to use the Chow group, the Chern class in cohomology is not sufficient.) Let its degree be m .

Fix a base point $0 \in A$. This defines a map from the Chow group of 0-cycles to $(A, 0)$; let $\alpha(\tilde{c}_1(L)^d)$ denote the image.

Finally let $T \subset A$ be the set of points $t \in A$ such that $m \cdot t = \alpha(\tilde{c}_1(L)^d)$. This T is a translate of the subgroup of m -torsion points. As before, the key point is that T is independent of the choice of the base point $0 \in A$. Indeed, if we change 0 be a translation by $c \in A$ then $\alpha(\tilde{c}_1(L)^d)$ is changed by translation by $m \cdot c$ so T is changed by translation by c .

Furthermore, if (A_b, L_b) is a family of polarized Abelian torsors that varies analytically (or algebraically) with b then $T_b \subset A_b$ is a family of subschemes that also vary analytically (or algebraically) with b . Thus we obtain the following.

Let $g : X \rightarrow B$ be a smooth, projective morphism whose fibers are Abelian torsors. Then there is a closed subscheme $T \subset X$ such that $g|_T : T \rightarrow B$ is étale and every fiber $T_b \subset A_b$ is a translate of the subgroup of m -torsion points (where $\deg T/B = m^{2d}$).

Lemma 46. *Let $g_i : X_i \rightarrow B$ be projective fiber spaces and $\phi : X_1 \dashrightarrow X_2$ a rational map. Assume the following.*

- (1) *There are no g_i -exceptional divisors.*
- (2) *A divisor on X_2 is \mathbb{Q} -Cartier iff its restriction to the generic fiber of g_2 is \mathbb{Q} -Cartier. (This holds trivially if X_2 is \mathbb{Q} -factorial.)*
- (3) *Every curve $C \subset X_2$ contracted by g_2 is \mathbb{Q} -homologous to a curve in a general fiber.*
- (4) *ϕ induces an isomorphism of the generic fibers of the g_i .*
- (5) *There are closed subsets $Z_i \subset X_i$ such that $\text{codim}_{X_i} Z_i \geq 2$ and ϕ induces an isomorphism $X_1 \setminus Z_1 \cong X_2 \setminus Z_2$.*

Then ϕ is an isomorphism.

Proof. Let $H_1 \subset X_1$ be a g_1 -ample divisor and $H_2 \subset X_2$ its birational transform. It follows from assumption (2) and (4) that H_2 is \mathbb{Q} -Cartier and from (3) that it is g_2 -ample. Thus (5) and a lemma of Matsusaka–Mumford [MM64] implies that ϕ is an isomorphism. (See [KSC04, 5.6] or [Kol10, Exrc.75] for the variant used here.) \square

47 (F-theory examples). Let X be a smooth, projective variety and $g : X \rightarrow B$ a relatively minimal elliptic fiber space with a section $\sigma : B \rightarrow X$. Since X is smooth, so is B .

Assume that $\Delta_{X/S} = 0$. Then, by (38), it can have only multiple smooth fibers at codimension 1 points, but then the section shows that there are no multiple fibers. Thus there is an open subset $B^0 \subset B$ such that $\text{codim}_B(B \setminus B^0) \geq 2$ and $X^0 \rightarrow B^0$ is a fiber bundle with fiber a pointed elliptic curve $(E, 0)$. Thus X^0 is given by the data

$$(B^0, E, \rho : \pi_1(B^0) \rightarrow \text{Aut}(E, 0)).$$

Note that $\pi_1(B^0) = \pi_1(B)$ since B is smooth and $\text{codim}_B(B \setminus B^0) \geq 2$. Thus X is birational to a fiber bundle $g' : X' \rightarrow B$ given by the data

$$(B, E, \rho : \pi_1(B) \rightarrow \text{Aut}(E, 0)).$$

All the fibers of g' are elliptic curves but the exceptional locus of a flip or a flop is always covered by rational curves (cf. [Kol96, VI.1.10]). Thus in fact $X \cong X'$

hence $g : X \rightarrow B$ is a locally trivial fiber bundle. The image of the monodromy representation $\rho : \pi_1(B) \rightarrow \text{Aut}(E, 0)$ is usually $\mathbb{Z}/2$, but for elliptic curves with extra automorphisms it can also be $\mathbb{Z}/3, \mathbb{Z}/4$ or $\mathbb{Z}/6$.

It is easy to write down examples where $K_X \sim 0$ and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$. However, $\pi_1(X)$ is always infinite, so such an X can not be a “true” Calabi–Yau manifold.

By (39), if $H^2(X, \mathcal{O}_X) = 0$ then every small deformation of X is obtained by deforming B and, if the image of ρ is $\mathbb{Z}/2$, also deforming E .

9. EXAMPLES

The first example is an elliptic Calabi–Yau surface with quotient singularities that has a flat smoothing which is neither Calabi–Yau nor elliptic.

Example 48. We start with a surface S_F^* which is the quotient of the square of the Fermat cubic curve by $\mathbb{Z}/3$:

$$S_F^* \cong (u_1^3 = v_1^3 + w_1^3) \times (u_2^3 = v_2^3 + w_2^3) / \frac{1}{3}(1, 0, 0; 1, 0, 0).$$

To describe the deformation, we need a different representation of it.

In \mathbb{P}^3 consider two lines $L_1 = (x_0 = x_1 = 0)$ and $L_2 = (x_2 = x_3 = 0)$. The linear system $|\mathcal{O}_{\mathbb{P}^2}(2)(-L_1 - L_2)|$ is spanned by the 4 reducible quadrics $x_i x_j$ for $i \in \{0, 1\}$ and $j \in \{2, 3\}$. They satisfy a relation $(x_0 x_2)(x_1 x_3) = (x_0 x_3)(x_1 x_2)$. Thus we get a morphism

$$\pi : B_{L_1+L_2} \mathbb{P}^3 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

which is a \mathbb{P}^1 -bundle whose fibers are the birational transforms of lines that intersect both of the L_i .

Let $S \subset \mathbb{P}^3$ be a cubic surface such that $\mathbf{p} := S \cap (L_1 + L_2)$ is 6 distinct points. Then we get $\pi_S : B_{\mathbf{p}} S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

In general, none of the lines connecting 2 points of \mathbf{p} is contained in S . Thus in this case π_S is a finite triple cover.

Both of the lines L_i determine an elliptic pencil on $B_{\mathbf{p}} S$ but if we move the 6 points \mathbf{p} into general position, we lose both elliptic pencils.

At the other extreme we have the Fermat-type surface

$$S_F := (x_0^3 + x_1^3 = x_2^3 + x_3^3) \subset \mathbb{P}^3.$$

We can factor both sides and write its equation as $m_1 m_2 m_3 = n_1 n_2 n_3$. The 9 lines $L_{ij} := (m_i = n_j = 0)$ are all contained in S_F . Let $L'_{ij} \subset B_{\mathbf{p}} S_F$ denote their birational transforms. Then the self-intersections $(L'_{ij} \cdot L'_{ij})$ equal -3 and π_{S_F} contracts these 9 curves L'_{ij} . Thus the Stein factorization of π_{S_F} gives a triple cover $S_F^* \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and S_F^* has 9 singular points of type $\mathbb{A}^2 / \frac{1}{3}(1, 1)$. We see furthermore that

$$-3K_{S_F} \sim \sum_{ij} L_{ij} \quad \text{and} \quad -3K_{B_{\mathbf{p}} S_F} \sim \sum_{ij} L'_{ij}.$$

Thus $-3K_{S_F^*} \sim 0$.

To see that this is the same S_F^* , note that the morphism of the original S_F^* to $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$(u_1 : v_1 : w_1) \times (u_2 : v_2 : w_2) \mapsto (v_1 : w_1) \times (v_2 : w_2)$$

and the rational map to the cubic surface is given by

$$(u_1 : v_1 : w_1) \times (u_2 : v_2 : w_2) \mapsto (v_2 u_1 u_2^2 : u_1 u_2^2 : v_1 u_2^3 : u_2^3).$$

Varying S gives a flat deformation whose central fiber is S_F^* , a surface with quotient singularities and torsion canonical class and whose general fiber is a cubic surface blown up at 6 general points, hence rational and without elliptic pencils.

The next example gives local models of generically isotrivial elliptic orbibundles that have a crepant resolution.

Example 49. Let $Z \subset \mathbb{P}^N$ be an anticanonically embedded Fano variety and $X \subset \mathbb{A}_{\mathbf{x}}^{N+1}$ the cone over Z . Let $0 \in E$ be an elliptic curve with a marked point. Consider the elliptic fiber space

$$Y := X \times E/(-1, -1) \rightarrow X/(-1).$$

We claim that Y has a crepant resolution.

First we blow up the vertex of X . We get $B_0X \rightarrow X$ with exceptional divisor $F \cong Z$. Note further that $B_0X \rightarrow X$ is crepant. The involution lifts to $B_0X \times E/(-1, -1)$. The fixed point set of this action is $F \times \{0\}$; a smooth subvariety of codimension 2. Thus $B_0X \times E/(-1, -1)$ is resolved by blowing up the singular locus.

The next example shows that for surfaces with normal crossing singularities, a deformation may lose the elliptic structure.

Example 50. Let $S \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a smooth surface of bi-degree $(1, 3)$. The first projection $\pi : S \rightarrow \mathbb{P}^1$ is an elliptic fiber space. The other projection $\tau : S \rightarrow \mathbb{P}^2$ exhibits it as the blow-up of \mathbb{P}^2 at 9 base points of an elliptic pencil. Let $F_1, \dots, F_9 \subset S$ denote the 9 exceptional curves. Thus S is an elliptic dP_9 . In particular, specifying $\pi : S \rightarrow \mathbb{P}^1$ plus a fiber of π is equivalent to a pair $(E \subset \mathbb{P}^2)$ plus 9 points $P_1, \dots, P_9 \in E$ such that $P_1 + \dots + P_9 \sim \mathcal{O}_{\mathbb{P}^2}(3)|_E$. The elliptic pencils are given by $\pi^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \tau^* \mathcal{O}_{\mathbb{P}^2}(3)(-F_1 - \dots - F_9)$.

Let us now vary the points on E in a family $P_i(t) : t \in \mathbb{C}$. The line bundle giving the elliptic pencil deforms as $\tau^* \mathcal{O}_{\mathbb{P}^2}(3)(-F_1(t) - \dots - F_9(t))$ but the elliptic pencil deforms only if $P_1(t) + \dots + P_9(t) \sim \mathcal{O}_{\mathbb{P}^2}(3)|_E$ holds for every t .

Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a smooth 3-fold of bi-degree $(1, 3)$. The first projection $\pi : X \rightarrow \mathbb{P}^2$ is an elliptic fiber space.

If $C \subset \mathbb{P}^2$ is a conic, its preimage $X_C \rightarrow C$ is an elliptic K3 surface. If C is general then X_C is smooth.

If $C = L_1 \cup L_2$ is a pair of general lines then $X_C = S_1 \cup S_2$ is a singular K3 surface which is a union of 2 smooth dP_9 that intersect along a smooth elliptic curve E .

We can thus think of X_C as obtained from two pairs $(E^i \subset \mathbb{P}^2)$ ($i = 1, 2$) with an isomorphism $\phi : E^1 \rightarrow E^2$ by blowing up 9 points $P_j^i \subset E^i$ ($j = 1, \dots, 9$) and gluing the resulting surfaces along the birational transforms of E^1 and E^2 .

Let us now vary the points on both curves $P_i^1(t)$ and $P_i^2(t)$. We get two families $S_1(t), S_2(t)$ and this induces a deformation $X_C(t) = S_1(t) \cup S_2(t)$.

Although the line bundle $\pi^* \mathcal{O}_C(1)$ giving the elliptic pencil $X_C \rightarrow C$ deforms on both of the $S_i(t)$, in general we do not get a line bundle on $X_C(t)$ unless

$$P_1^1(t) + \dots + P_9^1(t) \sim \phi^*(P_1^2(t) + \dots + P_9^2(t))$$

holds for every t . We can thus arrange that $\pi^* \mathcal{O}_C(1)$ deforms along $X_C(t)$ but we lose the elliptic pencil.

10. GENERAL CONJECTURES

A straightforward generalization of Conjecture 4 is the following, cf. [Ogu93] and [MP97, Lect.10].

Conjecture 51 (Strong abundance for Calabi–Yau manifolds). *Let X be a Calabi–Yau manifold and $L \in H^2(X, \mathbb{Q})$ a $(1, 1)$ -class such that $(L \cdot C) \geq 0$ for every algebraic curve $C \subset X$. Then there is a unique morphism with connected fibers $g : X \rightarrow B$ onto a normal variety B and an ample $L_B \in H^2(B, \mathbb{Q})$ such that $L = g^*L_B$.*

The usual abundance conjecture assumes that L is effective, but this may not be necessary.

One expects that (51) gets harder as the dimension of B decreases. The easiest case, when $\dim B = \dim X - 1$ corresponds to Questions 1–2.

From the point of view of higher dimensional birational geometry, it is natural to consider a more general setting.

A *log Calabi–Yau fiber space* is a proper morphism with connected fibers $g : (X, \Delta) \rightarrow B$ onto a normal variety where (X, Δ) is klt (or possibly lc) and $(K_X + \Delta)|_{X_g} \sim_{\mathbb{Q}} 0$ where $X_g \subset X$ is a general fiber.

Let (X, Δ) be a proper klt pair such that $K_X + \Delta$ is nef and $g : (X, \Delta) \rightarrow B$ a relatively minimal Calabi–Yau fiber space. Let L_B be an ample \mathbb{Q} -divisor on B and set $L := g^*L_B$. Then $L - \epsilon(K_X + \Delta)$ is nef for $0 \leq \epsilon \ll 1$. The converse fails in some rather simple cases, for instance when $X = B \times E$ for an elliptic curve E and we twist L by a degree zero non-torsion line bundle on E .

It is natural to expect that the above are essentially the only exceptions.

Conjecture 52. *Let (X, Δ) be a proper klt pair such that $K_X + \Delta$ is nef and $H^1(X, \mathcal{O}_X) = 0$. Let L be a Cartier divisor on X such that $L - \epsilon(K_X + \Delta)$ is nef for $0 \leq \epsilon \ll 1$.*

*Then there is a relatively minimal log Calabi–Yau fiber space structure $g : (X, \Delta) \rightarrow B$ and an ample \mathbb{Q} -divisor L_B on B such that $L \sim_{\mathbb{Q}} g^*L_B$.*

If $L - \epsilon(K_X + \Delta)$ is effective then (52) is implied by the Abundance Conjecture. Note also that (50) shows that (52) fails if (X, Δ) is log canonical.

Conjecture 53. *Let $g_0 : (X_0, \Delta_0) \rightarrow B_0$ be a log Calabi–Yau fiber space where (X_0, Δ_0) is a proper klt pair and $H^2(X_0, \mathcal{O}_{X_0}) = 0$.*

Let (X, Δ) be a klt pair and $h : (X, \Delta) \rightarrow (0 \in S)$ a flat proper morphism whose central fiber is (X_0, Δ_0) .

Then, after passing to an analytic or étale neighborhood of $0 \in S$, there is a proper, flat morphism $B \rightarrow (0 \in S)$ whose central fiber is B_0 such that g_0 extends to a log Calabi–Yau fiber space $g : (X, \Delta) \rightarrow B$.

54. Although (53) looks much more general than (31), it seems that Abelian fibrations comprise the only unknown case.

Indeed, let X_0, B_0 be projective varieties with rational singularities and $g_0 : X_0 \rightarrow B_0$ a morphism with connected general fiber F_0 . Assume that $H^1(F_0, \mathcal{O}_{F_0}) = 0$. Then $R^1(g_0)_*\mathcal{O}_{X_0}$ is a torsion sheaf. On the other hand, it is reflexive by [Kol86a, 7.8]. Thus $R^1(g_0)_*\mathcal{O}_{X_0} = 0$.

Let L_{B_0} be a sufficiently ample line bundle on B_0 and set $L_0 := g_0^*L_{B_0}$. Then $H^1(X_0, L_0) = 0$ by (21.1). Thus, if $h : X \rightarrow (0 \in S)$ is a deformation of X_0 such

that L_0 lifts to a line bundle L on X then every section of L_0 lifts to a section of L (after passing to an analytic or étale neighborhood of $0 \in S$). Thus (53) holds in this case.

Furthermore, the method of (30) suggests that the most difficult case is Abelian pencils over \mathbb{P}^1 .

Note also that it is easy to write down examples of Abelian Calabi–Yau fiber spaces $f : X \rightarrow B = \mathbb{P}^1$ such that $\mathrm{Hom}_B(\Omega_B, R^1 f_* \mathcal{O}_X) \neq 0$, thus (35) does not seem to be sufficient to prove (53).

55 (Log elliptic fiber spaces). As before, $g : (X, \Delta) \rightarrow B$ is a log elliptic fiber space iff $(L^{\dim X}) = 0$ but $(L^{\dim X-1}) \neq 0$. There are 3 cases to consider.

- (1) If $(L^{\dim X-1} \cdot \Delta) > 0$ then Riemann–Roch shows that $h^0(X, L^m)$ grows like $m^{\dim X-1}$ and we get (52) as in (10). In this case the general fiber of g is $F \cong \mathbb{P}^1$ and $(F \cdot \Delta) = 2$.
- (2) If $(L^{\dim X-1} \cdot \Delta) = 0$ but $(L^{n-2} \cdot \mathrm{td}_2(X)) > 0$ then the proof of (10) works with minor changes.
- (3) The hard and unresolved case is again when $(L^{\dim X-1} \cdot \Delta) = 0$ and $(L^{n-2} \cdot \mathrm{td}_2(X)) = 0$, so $\chi(X, L^m) = O(m^{\dim X-3})$.

Acknowledgments. I thank G. Di Cerbo, R. Donagi, O. Fujino, A. Langer, R. Lazarsfeld, K. Oguiso, Y.-C. Tu and C. Xu for helpful discussions, comments and references. Partial financial support was provided by the NSF under grant number DMS-07-58275.

REFERENCES

- [ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths, *Geometry of algebraic curves. Volume II*, Grundlehren der Mathematischen Wissenschaften, vol. 268, Springer, Heidelberg, 2011, With a contribution by Joseph Daniel Harris. MR 2807457 (2012e:14059)
- [AH11] Dan Abramovich and Brendan Hassett, *Stable varieties with a twist*, Classification of algebraic varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 1–38. MR 2779465
- [AK03] Carolina Araujo and János Kollár, *Rational curves on varieties*, Higher dimensional varieties and rational points (Budapest, 2001), Bolyai Soc. Math. Stud., vol. 12, Springer, Berlin, 2003, pp. 13–68. MR MR2011743 (2004k:14049)
- [Art69] M. Artin, *Algebraization of formal moduli. I*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 21–71. MR MR0260746 (41 #5369)
- [AW98] Marco Andreatta and Jarosław A. Wiśniewski, *On contractions of smooth varieties*, J. Algebraic Geom. **7** (1998), no. 2, 253–312. MR MR1620110 (99f:14021a)
- [BGS11] S. Boissière, O. Gabber, and O. Serman, *Sur le produit de variétés localement factorielles ou Q -factorielles*, ArXiv e-prints (2011).
- [BHPS12] B. Bhatt, W. Ho, Z. Patakfalvi, and C. Schnell, *Moduli of products of stable varieties*, ArXiv e-prints (2012).
- [BPV84] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 4, Springer-Verlag, Berlin, 1984. MR 749574 (86c:32026)
- [Bri10] Michel Brion, *Some basic results on actions of nonaffine algebraic groups*, Symmetry and spaces, Progr. Math., vol. 278, Birkhäuser Boston Inc., Boston, MA, 2010, pp. 1–20. MR 2562620 (2011c:14128)
- [Cat07] Fabrizio Catanese, *Q.E.D. for algebraic varieties*, J. Differential Geom. **77** (2007), no. 1, 43–75, With an appendix by Sönke Rollenske. MR 2344354 (2008k:14038)
- [CL10] J. I. Cogolludo-Agustin and A. Libgober, *Mordell–Weil groups of elliptic threefolds and the Alexander module of plane curves*, ArXiv e-prints (2010).

- [Del71] Pierre Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR MR0498551 (58 #16653a)
- [DG94] Igor Dolgachev and Mark Gross, *Elliptic threefolds. I. Ogg-Shafarevich theory*, J. Algebraic Geom. **3** (1994), no. 1, 39–80. MR 1242006 (95d:14037)
- [Don98] R. Y. Donagi, *ICMP lecture on Heterotic/F-theory duality*, ArXiv High Energy Physics - Theory e-prints (1998).
- [Fuj78] Takao Fujita, *On Kähler fiber spaces over curves*, J. Math. Soc. Japan **30** (1978), no. 4, 779–794. MR MR513085 (82h:32024)
- [Fuj11] Osamu Fujino, *On Kawamata's theorem*, Classification of algebraic varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 305–315. MR 2779478 (2012c:14011)
- [Ful98] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998. MR MR1644323 (99d:14003)
- [GHS03] Tom Graber, Joe Harris, and Jason Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67 (electronic). MR MR1937199 (2003m:14081)
- [GKKP11] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell, *Differential forms on log canonical spaces*, Publ. Math. Inst. Hautes Études Sci. (2011), no. 114, 87–169. MR 2854859
- [Gra91] Antonella Grassi, *On minimal models of elliptic threefolds*, Math. Ann. **290** (1991), no. 2, 287–301. MR 1109635 (92f:14013)
- [Gra93] ———, *The singularities of the parameter surface of a minimal elliptic threefold*, Internat. J. Math. **4** (1993), no. 2, 203–230. MR 1217381 (94h:14036)
- [Gra94] ———, *On a question of J. Kollár*, Classification of algebraic varieties (L'Aquila, 1992), Contemp. Math., vol. 162, Amer. Math. Soc., Providence, RI, 1994, pp. 209–214. MR 1272699 (95a:14036)
- [Gro62] Alexander Grothendieck, *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*, Secrétariat mathématique, Paris, 1962. MR MR0146040 (26 #3566)
- [Gro94] Mark Gross, *A finiteness theorem for elliptic Calabi-Yau threefolds*, Duke Math. J. **74** (1994), no. 2, 271–299. MR 1272978 (95c:14047)
- [Gro97] ———, *Elliptic three-folds. II. Multiple fibres*, Trans. Amer. Math. Soc. **349** (1997), no. 9, 3409–3468. MR 1401771 (98i:14036)
- [Har10] Robin Hartshorne, *Deformation theory*, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010. MR 2583634
- [HK10] Christopher D. Hacon and Sándor J. Kovács, *Classification of higher dimensional algebraic varieties*, Oberwolfach Seminars, vol. 41, Birkhäuser Verlag, Basel, 2010. MR 2675555 (2011f:14025)
- [HK11] Klaus Hulek and Remke Kloosterman, *Calculating the Mordell-Weil rank of elliptic threefolds and the cohomology of singular hypersurfaces*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 3, 1133–1179. MR 2918726
- [Hor76] Eiji Horikawa, *On deformations of holomorphic maps. III*, Math. Ann. **222** (1976), no. 3, 275–282. MR 0417458 (54 #5508)
- [HX12] Christopher Hacon and Chenyang Xu, *Existence of log canonical closures*, Inventiones Math. (2012), (to appear).
- [Kaw85a] Y. Kawamata, *Pluricanonical systems on minimal algebraic varieties*, Invent. Math. **79** (1985), no. 3, 567–588. MR MR782236 (87h:14005)
- [Kaw85b] Yujiro Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46. MR MR814013 (87a:14013)
- [KL09] János Kollár and Michael Larsen, *Quotients of Calabi-Yau varieties*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, Progr. Math., vol. 270, Birkhäuser Boston Inc., Boston, MA, 2009, pp. 179–211. MR 2641190 (2011c:14117)
- [Klo11] R. Kloosterman, *Cuspidal plane curves, syzygies and a bound on the MW-rank*, ArXiv e-prints (2011).
- [KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 1658959 (2000b:14018)

- [KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori, *Rationally connected varieties*, J. Algebraic Geom. **1** (1992), no. 3, 429–448. MR MR1158625 (93i:14014)
- [KN94] Yujiro Kawamata and Yoshinori Namikawa, *Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties*, Invent. Math. **118** (1994), no. 3, 395–409. MR MR1296351 (95j:32030)
- [Kod63] K. Kodaira, *On compact analytic surfaces. II, III*, Ann. of Math. (2) **77** (1963), 563–626; *ibid.* **78** (1963), 1–40. MR 0184257 (32 #1730)
- [Kol85] János Kollár, *Toward moduli of singular varieties*, Compositio Math. **56** (1985), no. 3, 369–398. MR MR814554 (87e:14009)
- [Kol86a] ———, *Higher direct images of dualizing sheaves. I*, Ann. of Math. (2) **123** (1986), no. 1, 11–42. MR 825838 (87c:14038)
- [Kol86b] ———, *Higher direct images of dualizing sheaves. II*, Ann. of Math. (2) **124** (1986), no. 1, 171–202. MR 847955 (87k:14014)
- [Kol92] János Kollár (ed.), *Flips and abundance for algebraic threefolds*, Société Mathématique de France, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992). MR 94f:14013
- [Kol93] ———, *Shafarevich maps and plurigenera of algebraic varieties*, Invent. Math. **113** (1993), no. 1, 177–215. MR 1223229 (94m:14018)
- [Kol95] ———, *Flatness criteria*, J. Algebra **175** (1995), no. 2, 715–727. MR 1339664 (96j:14010)
- [Kol96] ———, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180 (98c:14001)
- [Kol10] ———, *Exercises in the birational geometry of algebraic varieties*, Analytic and algebraic geometry, IAS/Park City Math. Ser., vol. 17, Amer. Math. Soc., Providence, RI, 2010, pp. 495–524. MR 2743822 (2012d:14015)
- [Kol12] ———, *Moduli of varieties of general type*, Handbook of Moduli (Gavril Farkas and Ian Morrison, eds.), Advanced Lectures in Mathematics, International Press (Somerville, MA), 2012, to appear.
- [Kol13] ———, *Singularities of the minimal model program*, Cambridge University Press, Cambridge, 2013, With the collaboration of S. Kovács (to appear).
- [KSC04] János Kollár, Karen E. Smith, and Alessio Corti, *Rational and nearly rational varieties*, Cambridge Studies in Advanced Mathematics, vol. 92, Cambridge University Press, Cambridge, 2004. MR 2062787 (2005i:14063)
- [Lai11] Ching-Jui Lai, *Varieties fibered by good minimal models*, Math. Ann. **350** (2011), no. 3, 533–547. MR 2805635 (2012e:14032)
- [Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I-II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 48–49, Springer-Verlag, Berlin, 2004. MR 2095471 (2005k:14001a)
- [LM75] D. Lieberman and D. Mumford, *Matsusaka’s big theorem*, Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), Amer. Math. Soc., Providence, R.I., 1975, pp. 513–530. MR MR0379494 (52 #399)
- [Mat68] T. Matsusaka, *Algebraic deformations of polarized varieties*, Nagoya Math. J. **31** (1968), 185–245; corrections, *ibid.* **33** (1968), 137; *ibid.* **36** (1968), 119. MR MR0258827 (41 #3473)
- [Mat72] ———, *Polarized varieties with a given Hilbert polynomial*, Amer. J. Math. **94** (1972), 1027–1077. MR MR0337960 (49 #2729)
- [Miy88] Yoichi Miyaoka, *On the Kodaira dimension of minimal threefolds*, Math. Ann. **281** (1988), no. 2, 325–332. MR MR949837 (89g:14028)
- [MM64] T. Matsusaka and D. Mumford, *Two fundamental theorems on deformations of polarized varieties*, Amer. J. Math. **86** (1964), 668–684. MR MR0171778 (30 #2005)
- [MP97] Y. Miyaoka and Th. Peternell, *Geometry of higher-dimensional algebraic varieties*, DMV Seminar, vol. 26, Birkhäuser Verlag, Basel, 1997.
- [Nak88] Noboru Nakayama, *The singularity of the canonical model of compact Kähler manifolds*, Math. Ann. **280** (1988), 509512.

- [Nak91] ———, *Elliptic fibrations over surfaces. I*, Algebraic geometry and analytic geometry (Tokyo, 1990), ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991, pp. 126–137. MR 1260943 (95c:14045)
- [Nak99] ———, *Projective algebraic varieties whose universal covering spaces are biholomorphic to \mathbf{C}^n* , J. Math. Soc. Japan **51** (1999), no. 3, 643–654. MR 1691481 (2000b:14019)
- [Nak02a] ———, *Global structure of an elliptic fibration*, Publ. Res. Inst. Math. Sci. **38** (2002), no. 3, 451–649. MR 1918053 (2003e:14006)
- [Nak02b] ———, *Local structure of an elliptic fibration*, Higher dimensional birational geometry (Kyoto, 1997), Adv. Stud. Pure Math., vol. 35, Math. Soc. Japan, Tokyo, 2002, pp. 185–295. MR 1929795 (2004b:14014)
- [Ogu93] Keiji Oguiso, *On algebraic fiber space structures on a Calabi-Yau 3-fold*, Internat. J. Math. **4** (1993), no. 3, 439–465, With an appendix by Noboru Nakayama. MR 1228584 (94g:14019)
- [Rei83] Miles Reid, *Minimal models of canonical 3-folds*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 131–180. MR MR715649 (86a:14010)
- [Ser01] Jean-Pierre Serre, *Espaces fibrés algébriques*, Séminaire C. Chevalley (1958), Documents Mathématiques, vol. 1, Soc. Math. France, Paris, 2001.
- [Ses63] C. S. Seshadri, *Quotient space by an abelian variety*, Math. Ann. **152** (1963), 185–194. MR 0164973 (29 #2264)
- [Ste76] Joseph Steenbrink, *Limits of Hodge structures*, Invent. Math. **31** (1975/76), no. 3, 229–257. MR MR0429885 (55 #2894)
- [Vaf96] C. Vafa, *Evidence for F-theory*, Nuclear Physics B **469** (1996), 403–415.
- [Wil89] P. M. H. Wilson, *Calabi-Yau manifolds with large Picard number*, Invent. Math. **98** (1989), no. 1, 139–155. MR 1010159 (90h:14044)
- [Wil94] ———, *The existence of elliptic fibre space structures on Calabi-Yau threefolds*, Math. Ann. **300** (1994), no. 4, 693–703. MR 1314743 (96a:14047)
- [Wil98] ———, *The existence of elliptic fibre space structures on Calabi-Yau threefolds. II*, Math. Proc. Cambridge Philos. Soc. **123** (1998), no. 2, 259–262. MR 1490200 (99f:14053)

Princeton University, Princeton NJ 08544-1000
 kollar@math.princeton.edu